ALGEBRAIC & GEOMETRIC MODELS FOR SPACE NETWORKING

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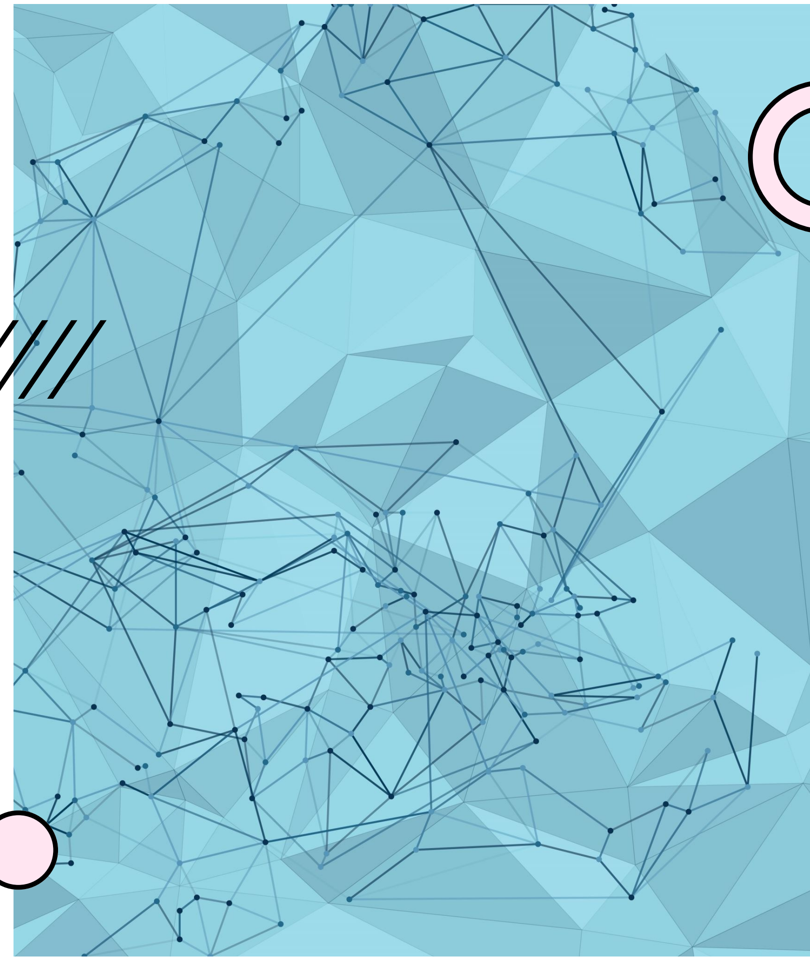
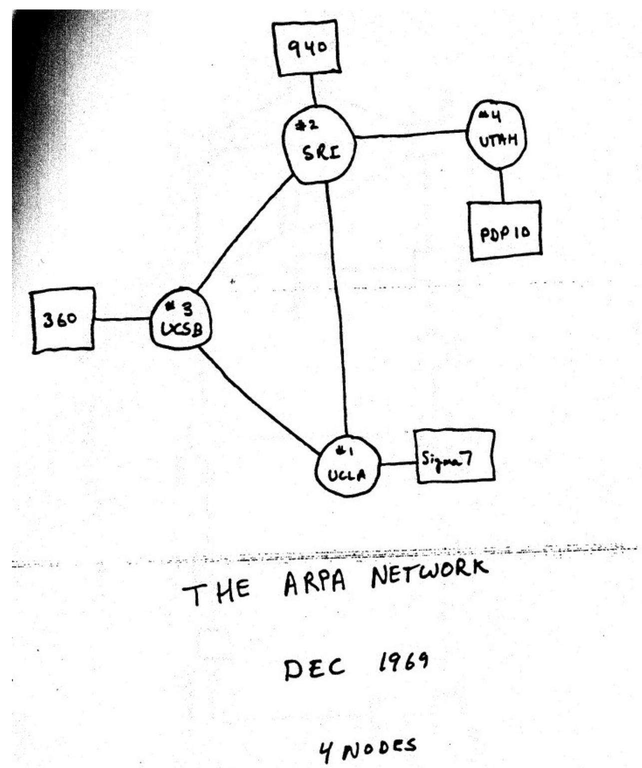
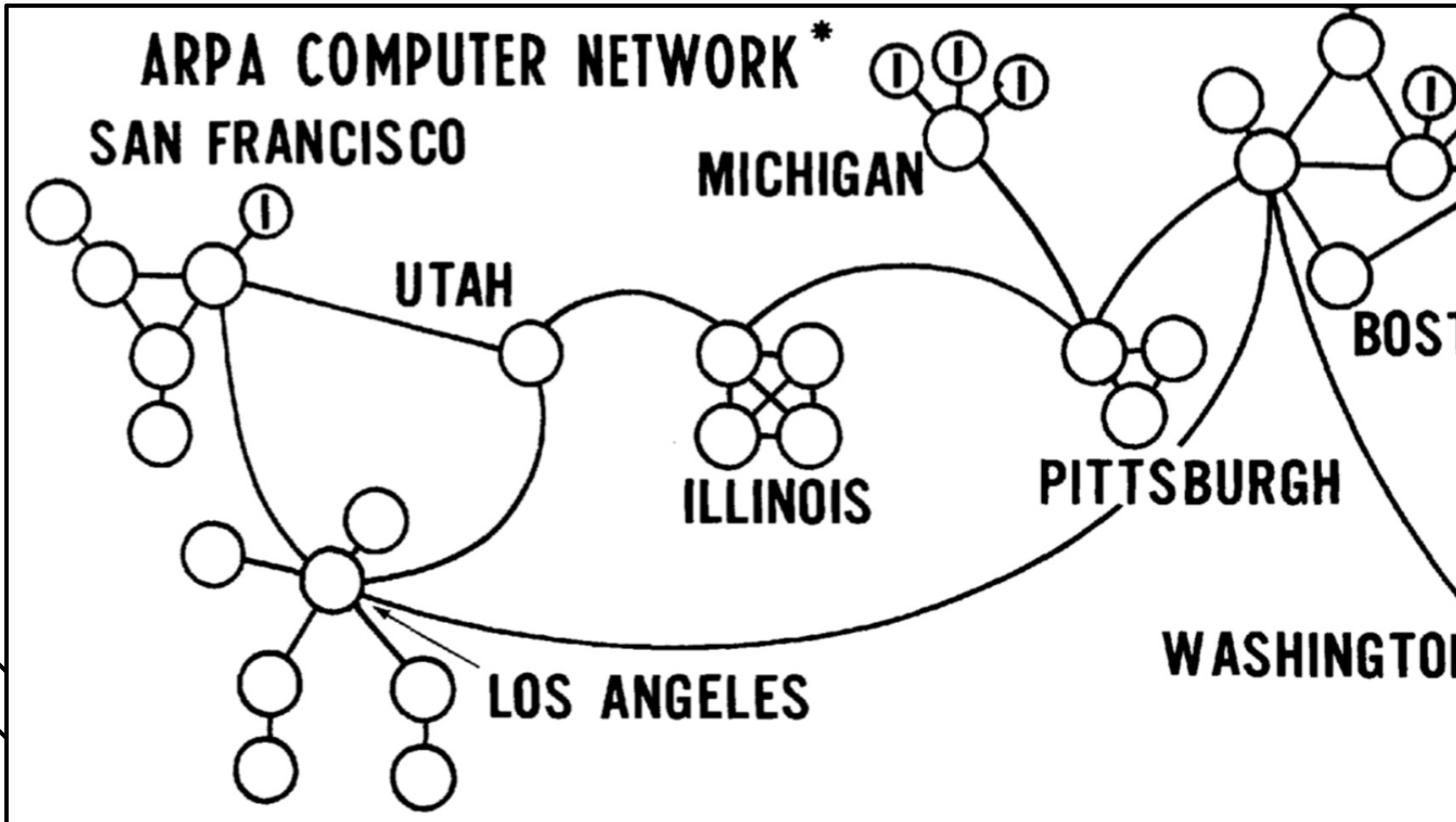
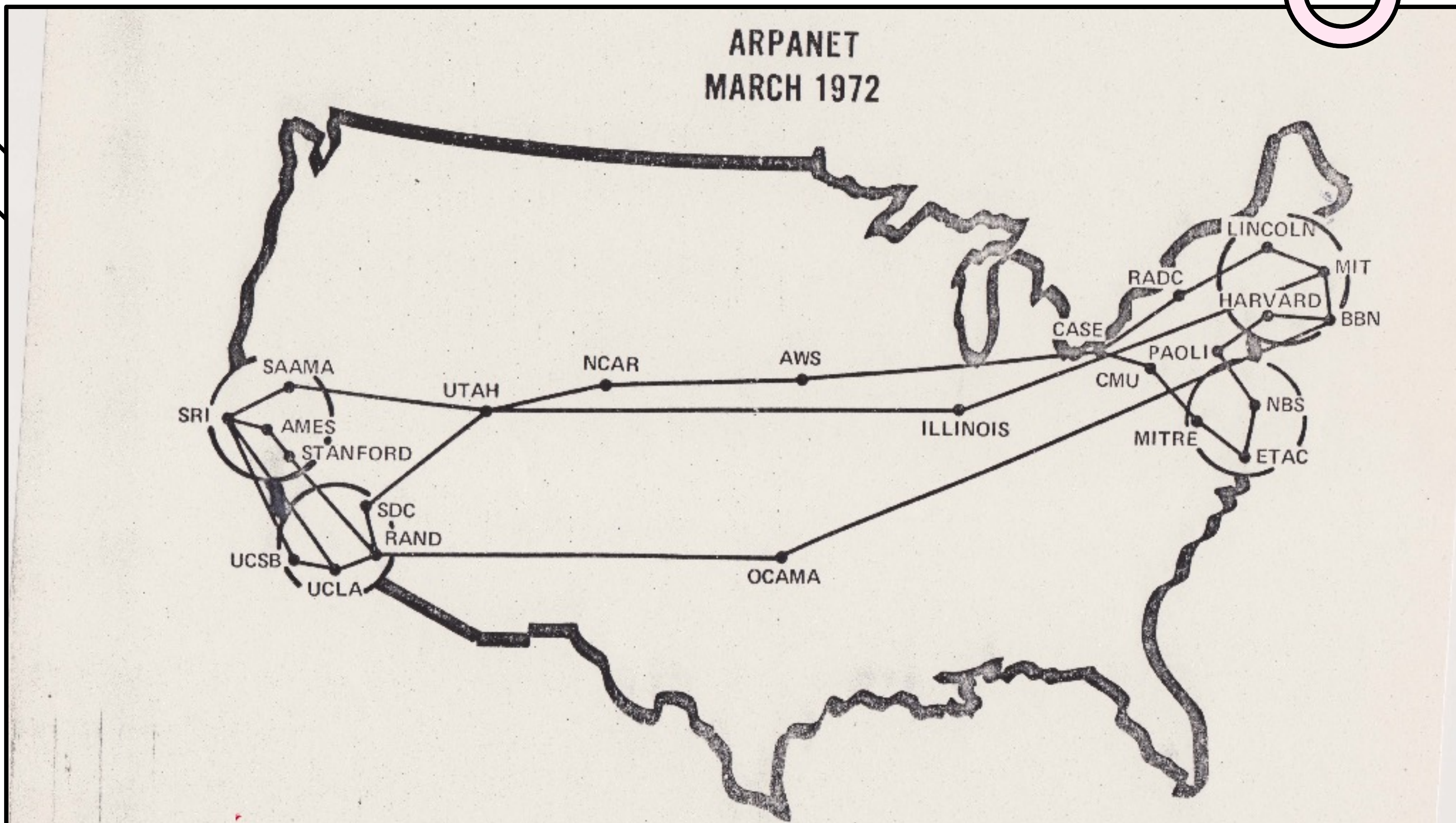


FIRST A BIT OF HISTORY...





Lawrence G. Roberts “Multiple Computer Networks and Intercomputer Communication” Oct. 2, 1967


ARPANET
MARCH 1972
LINCOLN
MIT
RADC
HARVARD
BBN
CASE
PAOLI
CMU
NBS
MITRE
ETAC
SAAMA
NCAR
AWS
UTAH
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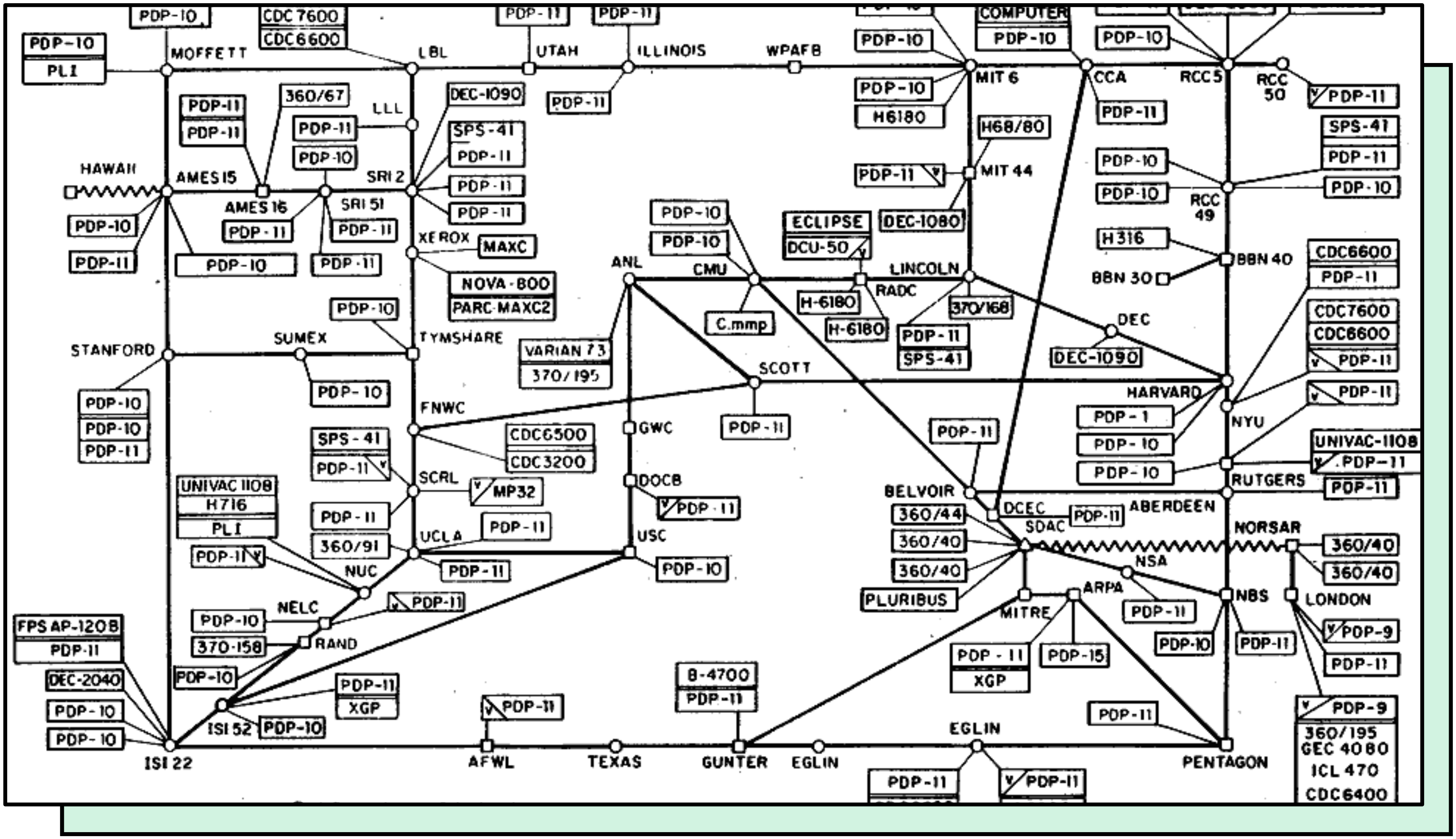
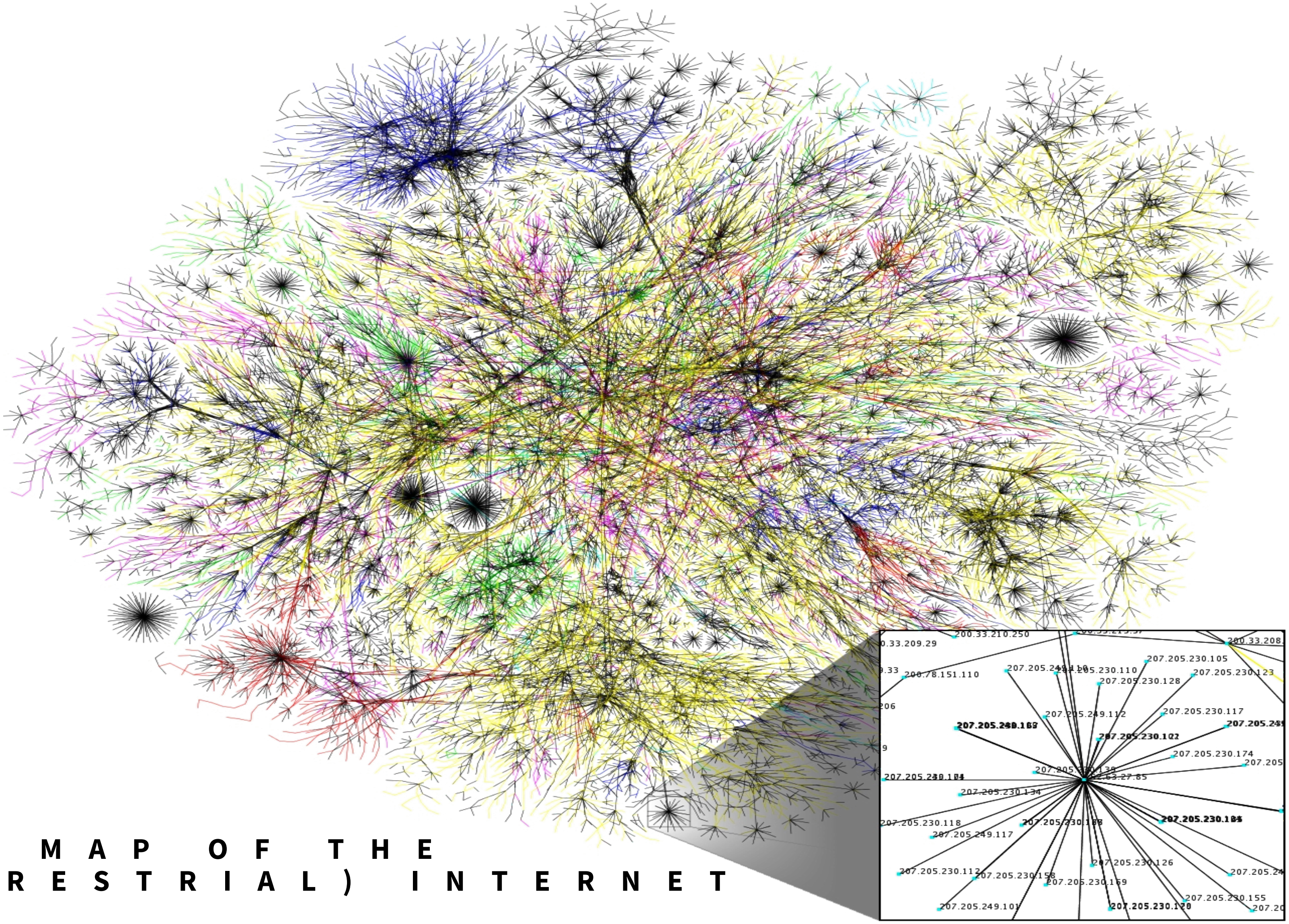
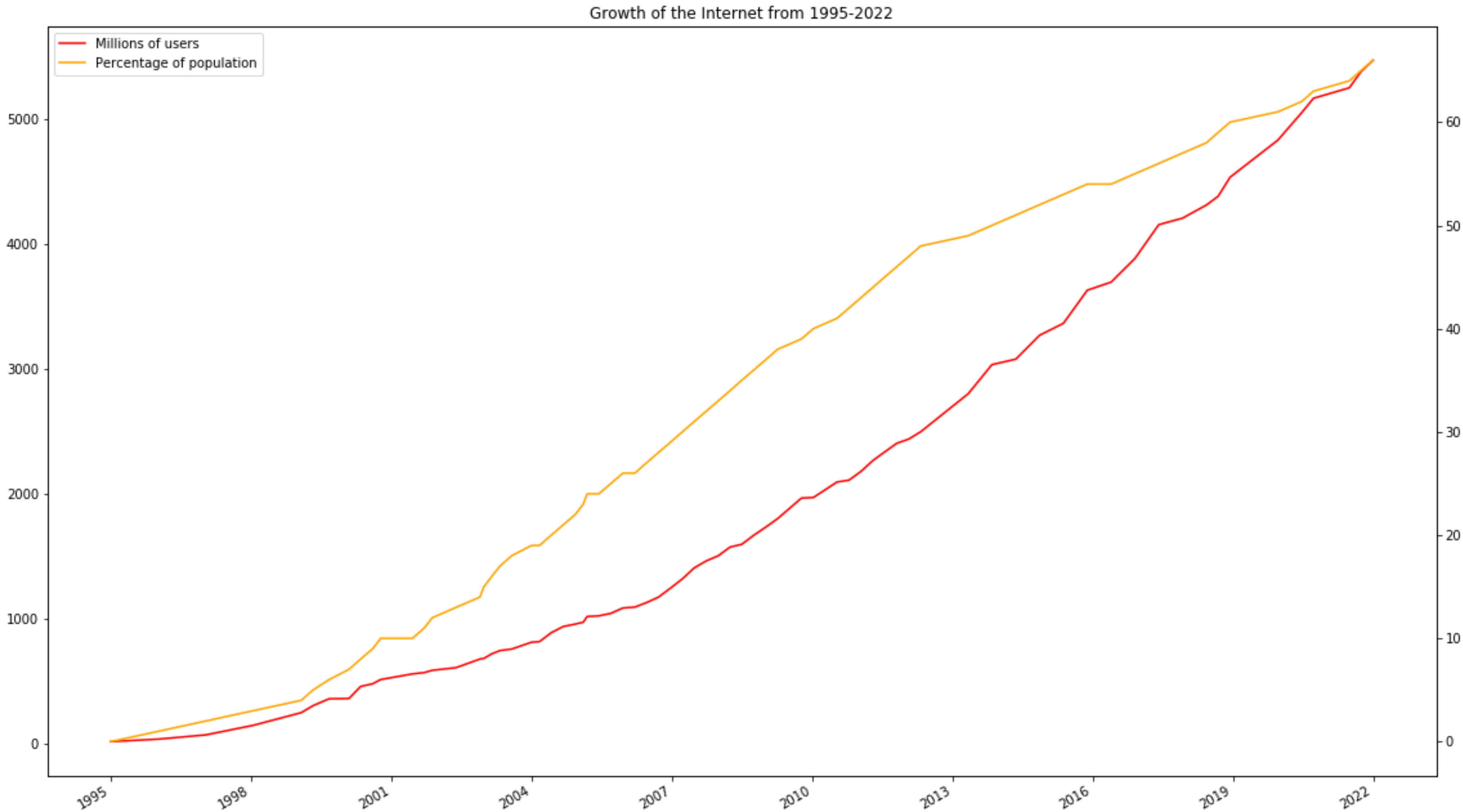
RAND
UCSB
UCLA
OCAMA



ARPANET Logical Map, Circa 1977 from https://www.sri.com/hoi/arpanet/



2005 MAP OF THE (TERRESTRIAL) INTERNET

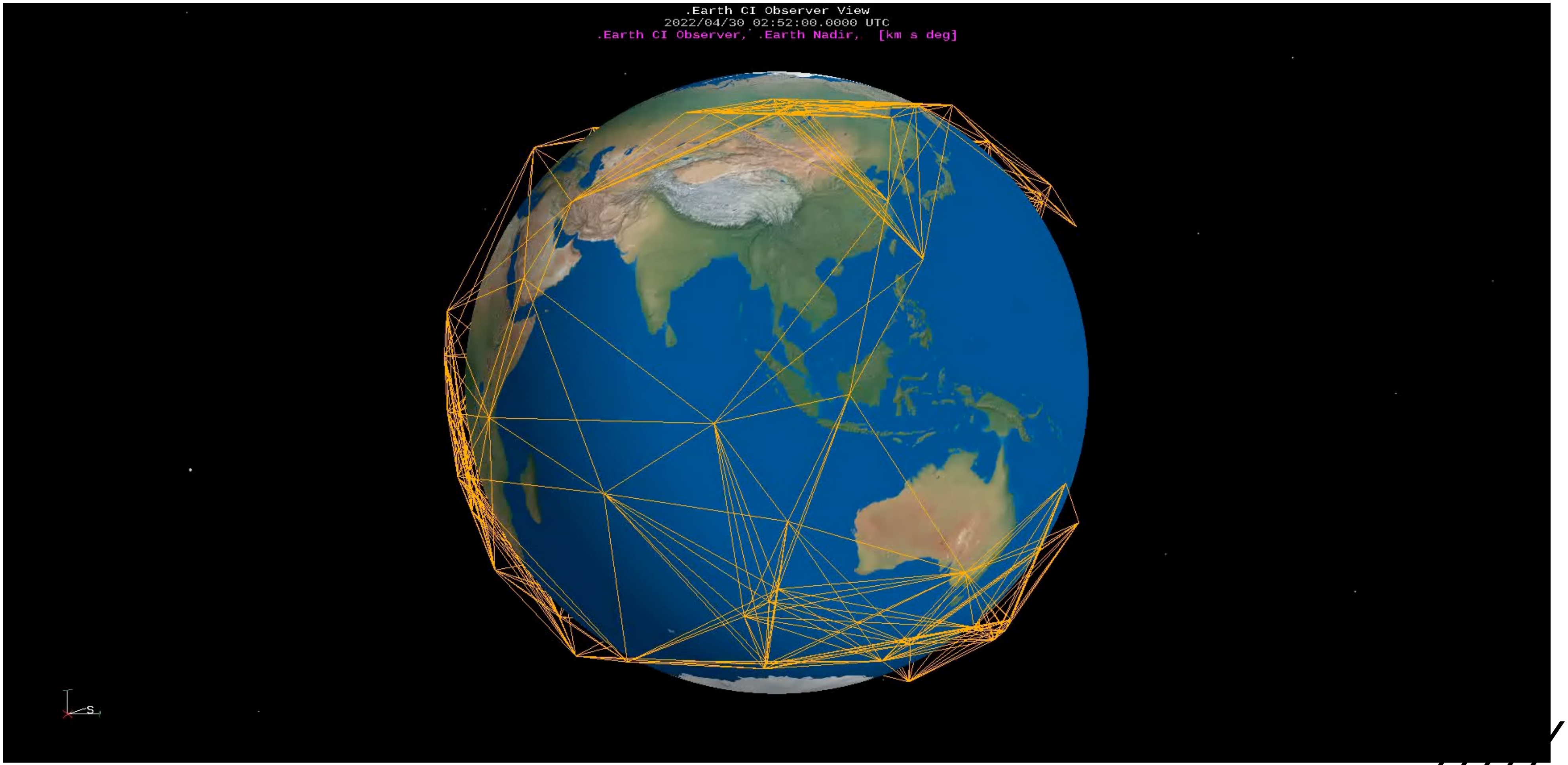

Growth of the Internet from 1995-2022
Millions of users
Percentage of population
0
1000
2000
3000
4000
5000
0
10
20
30
40
50
60
1995
1998
2001
2004
2007
2010
2013
2016
2019
2022

TRANSITION TO SPACE INTERNET

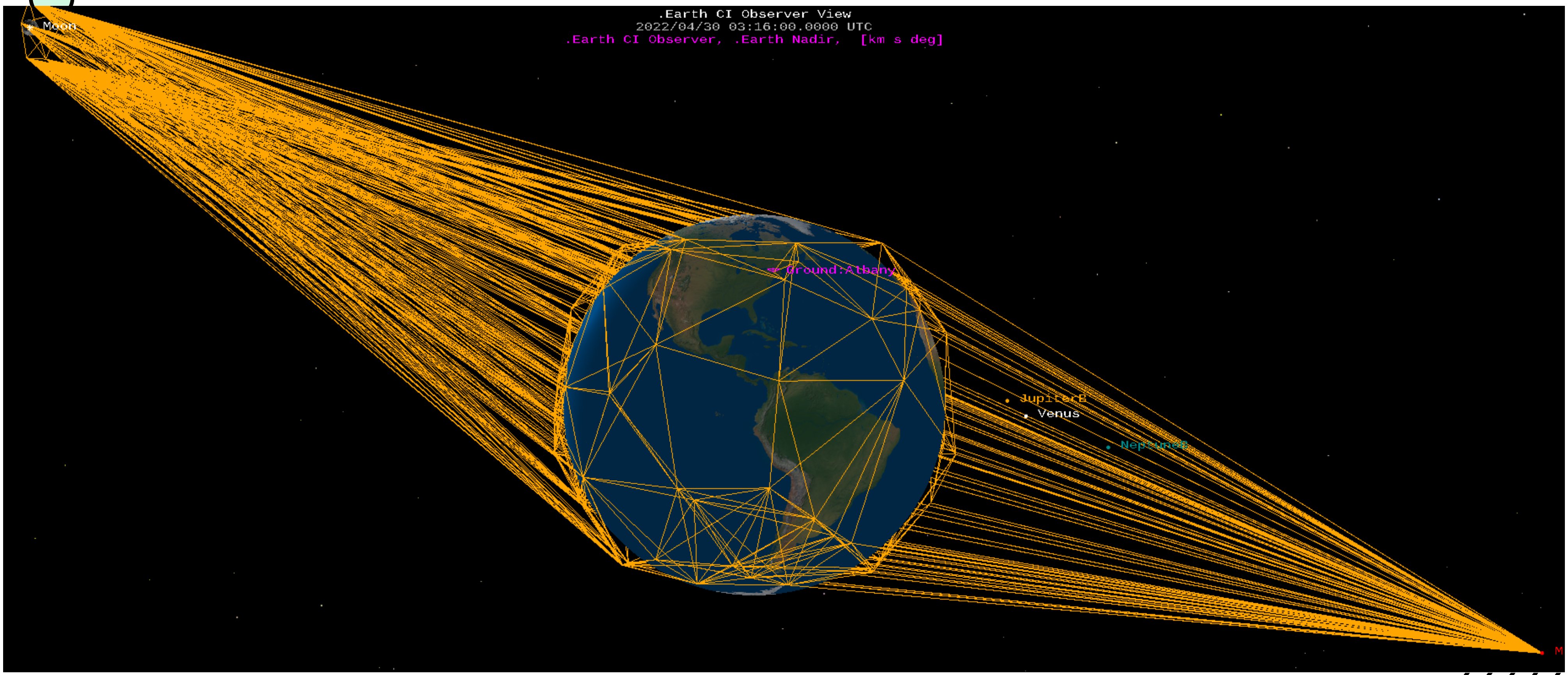
SATELLITE–BASED NETWORKS

8 0 S T A R L I N K N O D E S

1 0 0 S T A R L I N K N O D E S



S T A R L I N K + M O O N + M A R S

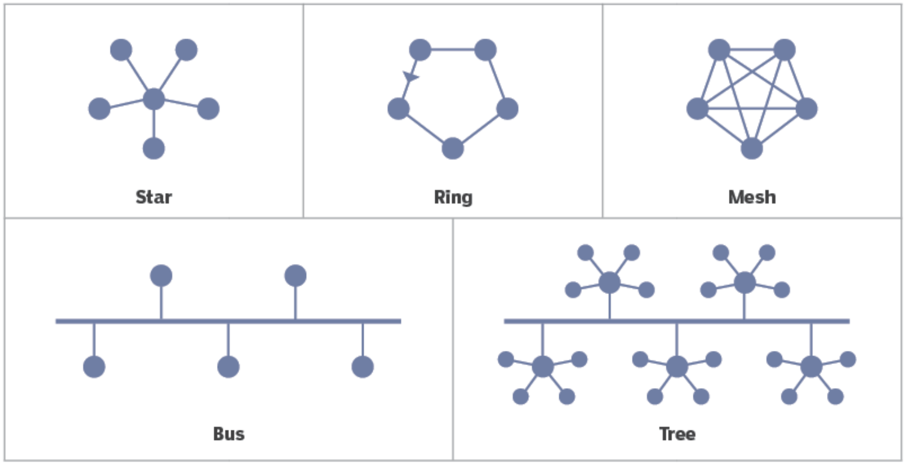


The Challenge of Space Networking:

Develop a *scalable*, *autonomous* routing protocol for solar system-wide internet

Aspects of this challenge:

1. Time-Varying Networks
2. Long One-Way Light Travel
3. Different network topologies "glued together”

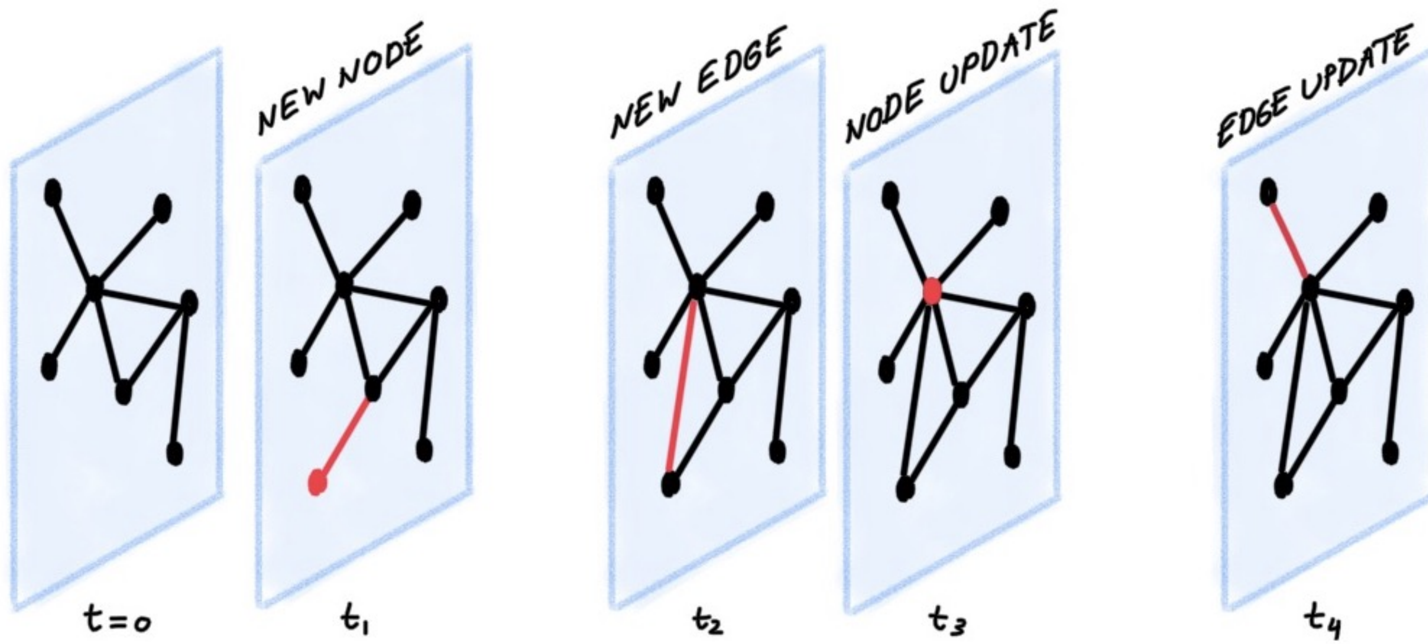


TIME-VARYING GRAPHS

SEMIRING MODELS
FOR
SPACE NETWORKING

What is a time-varying graph?

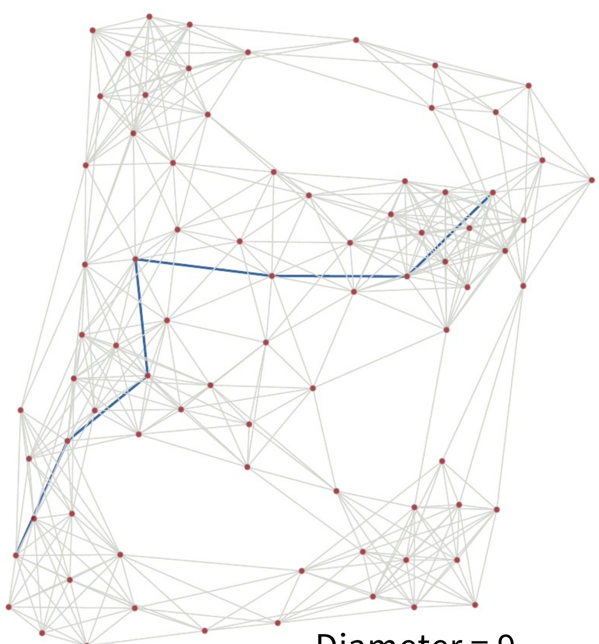
This is *not* the final answer!



From https://blog.twitter.com/engineering/en_us/topics/insights/2021/temporal-graph-networks

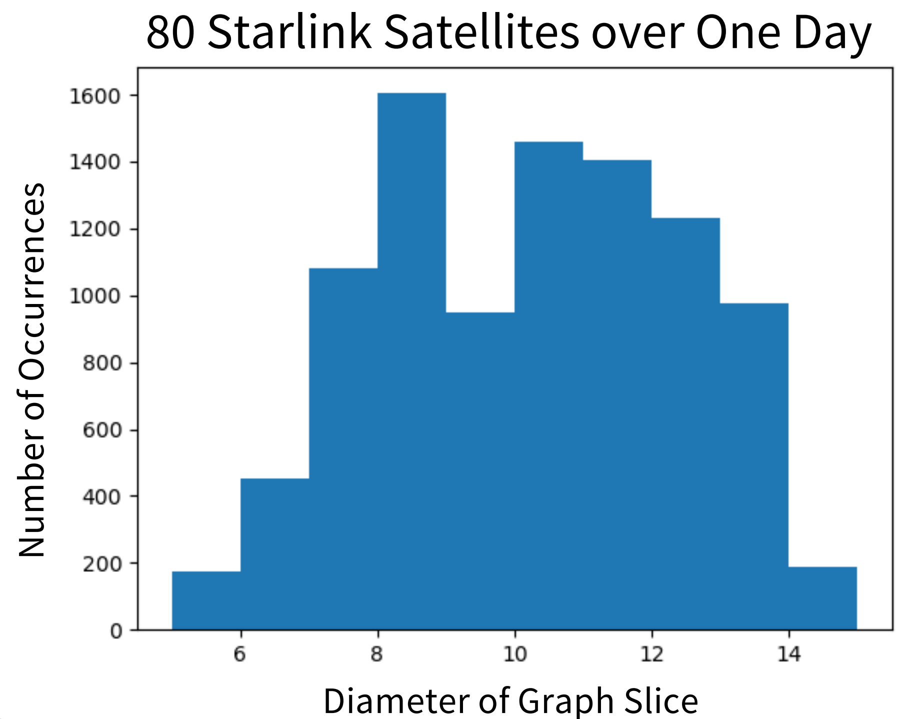
Graph Sequences

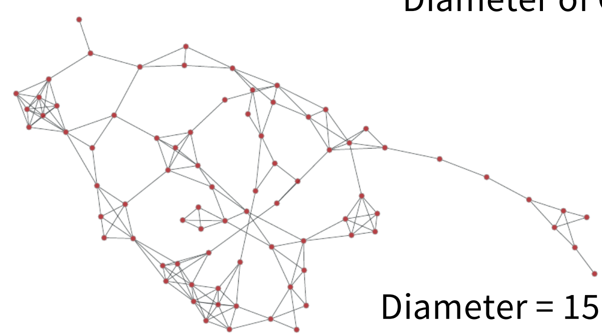
- No obvious temporal duration
- "Continuity" undefined
- Graphs studied "slice by slice", e.g. diameter statistics



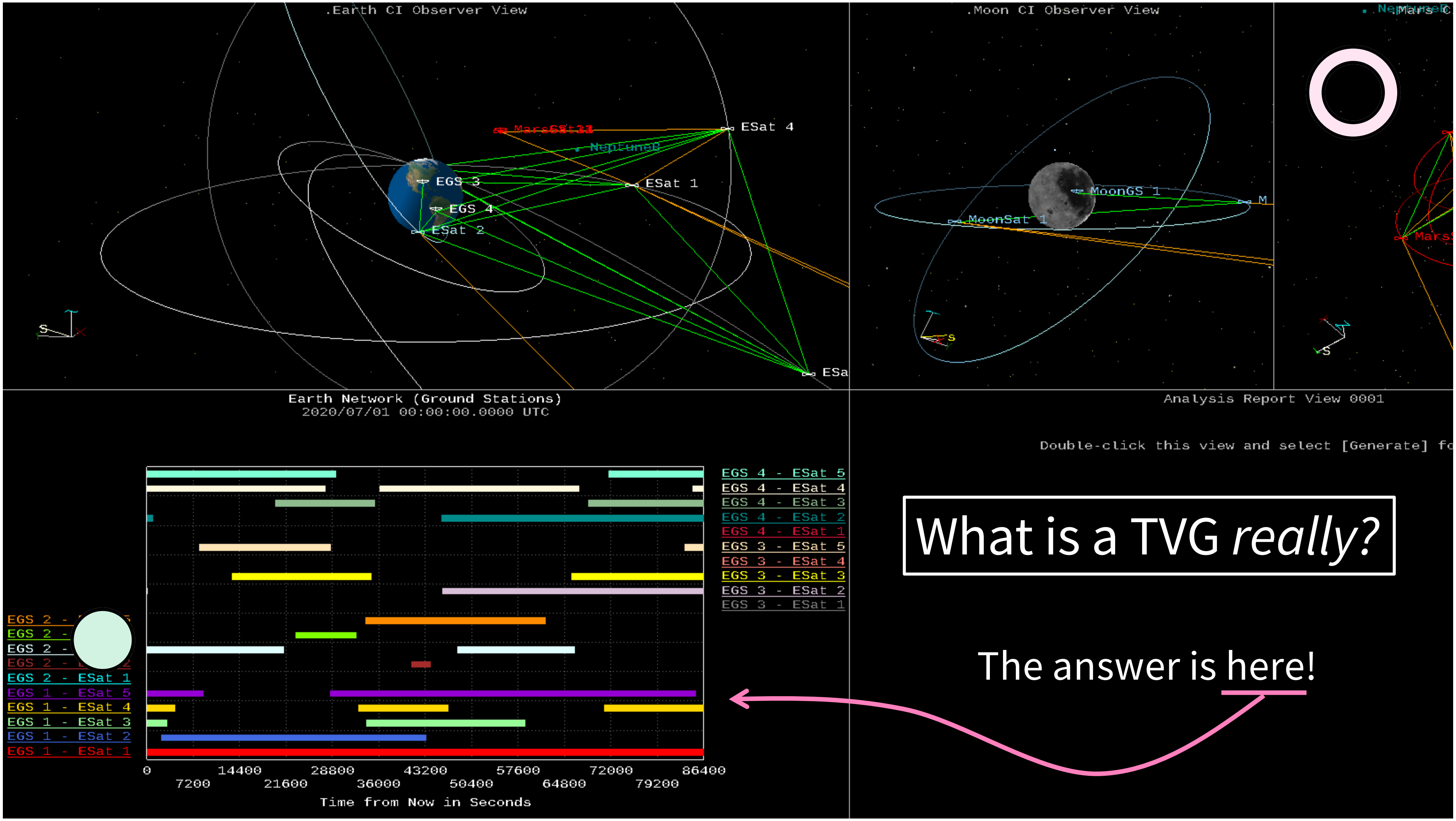
Diameter = 9

Diameter: the longest shortest path





Diameter = 15



What is a TVG *really?*

The answer is here!

Definition of a TVG

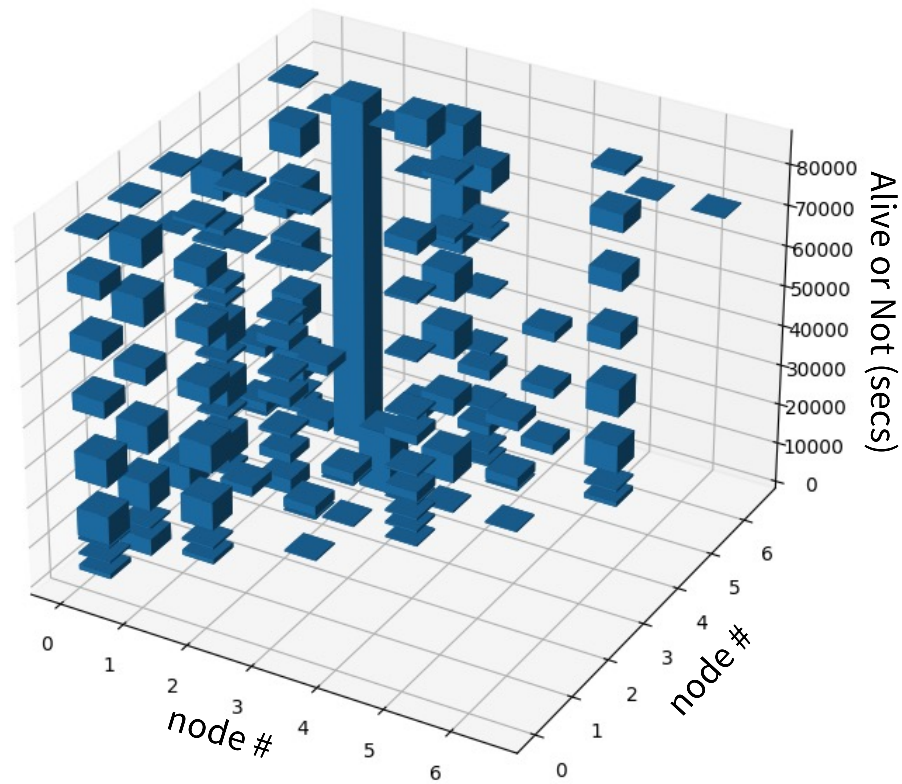
A **time-varying graph** (TVG) is a matrix of subsets of time

$$\boxed{M(i,j) \in \mathcal{P}(\mathbb{R})}$$

Or… The set of all subsets of time

$$\boxed{M : G^{op} \to \mathcal{P}(\mathbb{R})}$$

Cellular Cosheaf ⇔ vertex alive whenever edge is alive



WHY MATRICES?

ARITHMETIC FOR TVGS

Static Graphs are Boolean Matrices

What can you do with matrices? *Multiply them!*

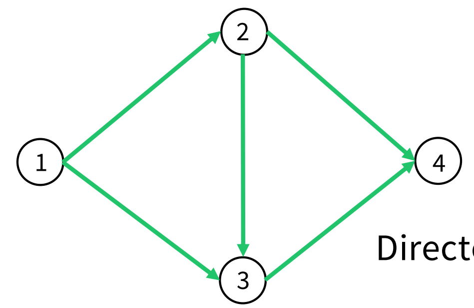
Semiring: a set S with **sum** $\oplus$ and **multiplication** $\otimes$ operation

Boolean Semiring

Elements	Operations
True = $\top$ = "1"	AND = $\wedge$ = $\otimes$
False = $\bot$ = "0"	OR = $\vee$ = $\oplus$

$$A = \begin{array}{c|c} & \text{To} \\ \text{From} & \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \bot & \top & \top & \bot \\ \bot & \bot & \top & \top \\ \bot & \bot & \bot & \top \\ \bot & \bot & \bot & \bot \end{bmatrix} \end{array} \end{array}$$

Adjacency Matrix



Directed Graph

Example row times column multiplication

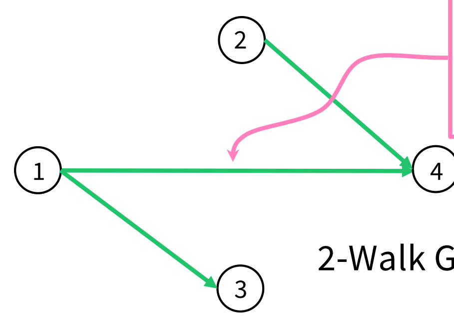
Bool $(\bot \wedge \bot) \vee (\top \wedge \top) \vee (\top \wedge \top) \vee (\bot \wedge \bot) = \top$

Name $\{\emptyset * \emptyset,\quad 12*24,\quad 13*34,\quad \emptyset * \emptyset\} = \{124, 134\}$

$$A^2 = \begin{bmatrix} \bot & \top & \top & \bot \\ \bot & \bot & \top & \top \\ \bot & \bot & \bot & \top \\ \bot & \bot & \bot & \bot \end{bmatrix} \begin{bmatrix} \bot & \top & \top & \bot \\ \bot & \bot & \top & \top \\ \bot & \bot & \bot & \top \\ \bot & \bot & \bot & \bot \end{bmatrix} = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \bot & \bot & \top & \top \\ \bot & \bot & \bot & \top \\ \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot \end{bmatrix} \end{array}$$

2-Walk Matrix

This edge witnesses a route from 1 to 4 of length two!



2-Walk Graph

Graphs and Semi-rings

What can you do with matrices? *Multiply them!*

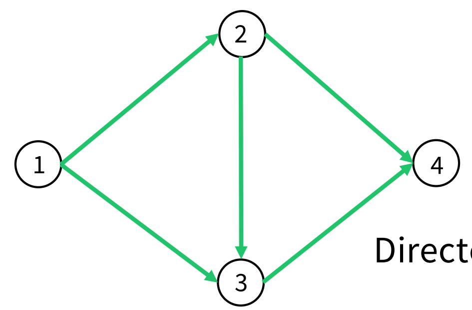
Semi-ring
is a set where you can **add** $\oplus$ and **multiply** $\otimes$

Boolean Semi-ring

Elements	Operations
True = $\top$ = "1"	AND = $\wedge$ = $\otimes$
False = $\bot$ = "0"	OR = $\vee$ = $\oplus$

$$A = \begin{array}{cc} & \text{To} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{From} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \bot & \top & \top & \bot \\ \bot & \bot & \top & \top \\ \bot & \bot & \bot & \top \\ \bot & \bot & \bot & \bot \end{bmatrix} \end{array}$$

Adjacency Matrix



Directed Graph

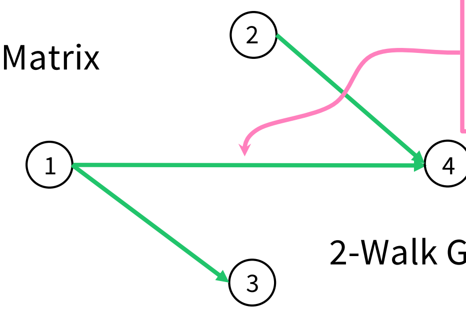
Example row times column multiplication

Bool $(\bot \wedge \bot) \vee (\top \wedge \top) \vee (\top \wedge \top) \vee (\bot \wedge \bot) = \top$

$$A^2 = \begin{bmatrix} \bot & \top & \top & \bot \\ \bot & \bot & \top & \top \\ \bot & \bot & \bot & \top \\ \bot & \bot & \bot & \bot \end{bmatrix} \begin{bmatrix} \bot & \top & \top & \bot \\ \bot & \bot & \top & \top \\ \bot & \bot & \bot & \top \\ \bot & \bot & \bot & \bot \end{bmatrix} = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \bot & \bot & \top & \top \\ \bot & \bot & \bot & \top \\ \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot \end{bmatrix} \end{array}$$

2-Walk Matrix

This edge witnesses a route from 1 to 4 of length two!



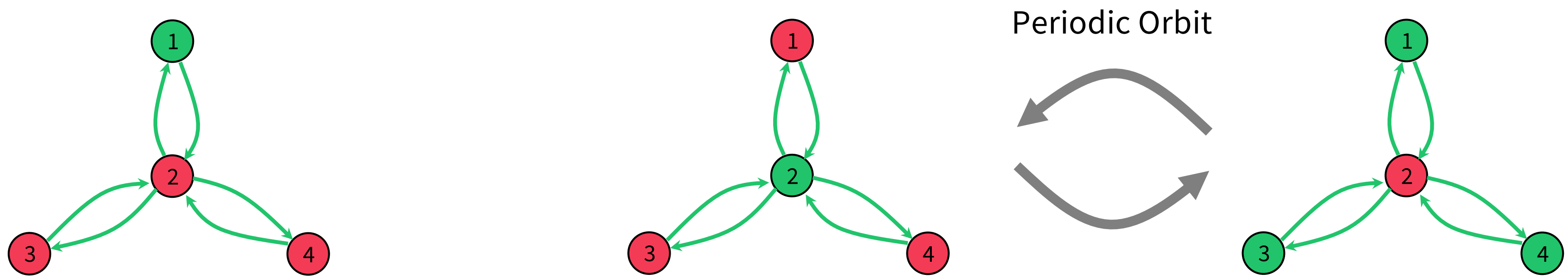
2-Walk Graph

Name (aka Path) Semi-ring

Concatenation of paths = $\otimes$
Disjoint union of paths = $\oplus$

Name $\{\emptyset * \emptyset, \quad 12{*}24, \quad 13{*}34, \quad \emptyset * \emptyset\} = \{124, 134\}$

Boolean Vector Multiplication



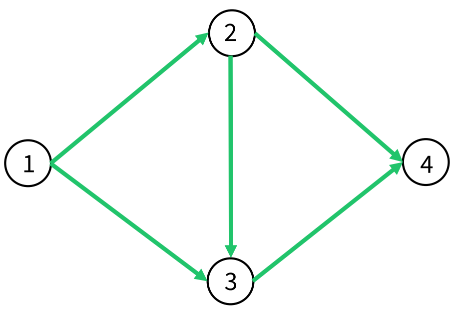
$$\underbrace{[\top \quad \bot \quad \bot \quad \bot]}_{v_0^T}\begin{bmatrix} \bot & \top & \bot & \bot \\ \top & \bot & \top & \top \\ \bot & \top & \bot & \bot \\ \bot & \top & \bot & \bot \end{bmatrix} \rightarrow \underbrace{[\bot \quad \top \quad \bot \quad \bot]}_{v_1^T}\begin{bmatrix} \bot & \top & \bot & \bot \\ \top & \bot & \top & \top \\ \bot & \top & \bot & \bot \\ \bot & \top & \bot & \bot \end{bmatrix} \rightarrow \underbrace{[\top \quad \bot \quad \top \quad \top]}_{v_2^T}\begin{bmatrix} \bot & \top & \bot & \bot \\ \top & \bot & \top & \top \\ \bot & \top & \bot & \bot \\ \bot & \top & \bot & \bot \end{bmatrix}$$

$v_0^T A = v_1^T$ $\quad$ $v_1^T A = v_2^T$ $\quad$ $v_2^T A = v_3^T$

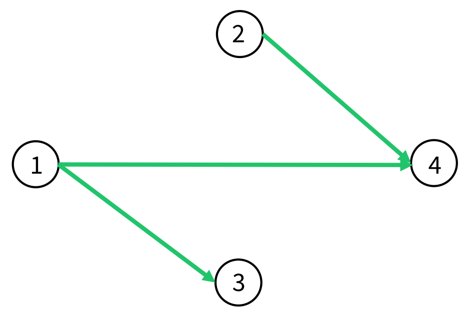
But! If we aggregate: $\boxed{v_0^T (I + A + A^2 + A^3 + \ldots) = [\top \quad \top \quad \top \quad \top]}$ $\boxed{\text{Boolean Footprint}}$

Every node is reached!

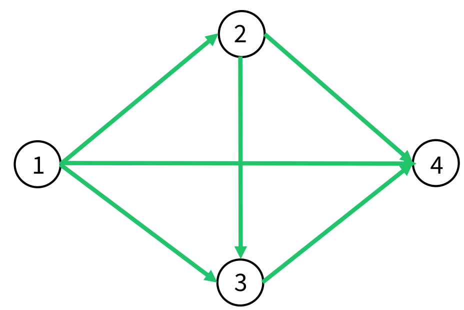
Accumulating Walks (For summarizing reach of a message)



Original Graph



2-Walk Graph



2-Cumulant Graph

$$A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \bot & \top & \top & \bot \\ \bot & \bot & \top & \top \\ \bot & \bot & \bot & \top \\ \bot & \bot & \bot & \bot \end{bmatrix} \end{array}$$

Adjacency Matrix
=
1-Walk Matrix

$$A^2 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \bot & \bot & \top & \top \\ \bot & \bot & \bot & \top \\ \bot & \bot & \bot & \bot \\ \bot & \bot & \bot & \bot \end{bmatrix} \end{array}$$

Adjacency Matrix
of 2-walk graph
=
2-Walk Matrix

$$(I + A + A^2) =: C_2 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \top & \top & \top & \top \\ \bot & \top & \top & \top \\ \bot & \bot & \top & \top \\ \bot & \bot & \bot & \top \end{bmatrix} \end{array}$$

2-Cumulant Matrix

$$\underbrace{[\top \ \ \bot \ \ \bot \ \ \bot]}_{v_0^T} \xrightarrow{C_2} \underbrace{[\top \ \ \top \ \ \top \ \ \top]}_{v_0^T + v_1^T + v_2^T}$$

The Kleene Star and its Cumulants

Assume A is an nxn matrix with entries in an idempotent semi-ring $R = (S, +, \times, 0, 1)$

This means a+a=a

Examples:
- $S=\{\top, \bot\}$ **Bool**
  $+=$OR and $\times=$ AND
  $0 = \bot$ and $1 = \top$
- $S=\mathcal{P}(\mathbb{R})$ **Lifetimes**
  $+=\cup$ and $\times=\cap$
  $0 = \emptyset$ and $1 = \mathbb{R}$

The *KLAY-nee* Star

$$A^* = \underbrace{I + A + A^2 + A^3}_{\text{The 3-Cumulant}} + \ldots = (I-A)^{-1}$$

Sketch of the Proof:
$(I+A)^2 = I + IA + AI + A^2$
$= I + A + A^2$

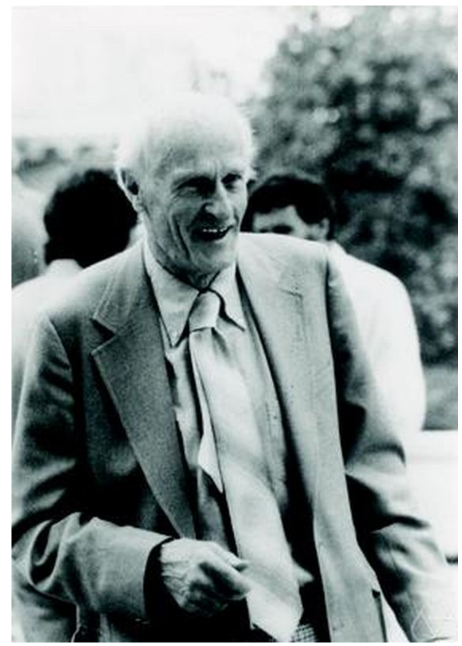
The **k-Cumulant** $C_k = I + A + A^2 + \ldots + A^k = (I+A)^k$

Under *Nice* Assumptions

$$\boxed{A^* = C_k = M^k \text{ for } k > n\text{-}1}$$

where M=I+A

First:
we define $a \preccurlyeq b$ if $a + b = b$
Second:
assume $w(\gamma) \preccurlyeq 1$ for any elementary cycle γ



Stephen Kleene (1909-1994)

The Kleene Star for Algebraic Network Theory

A is an n x n matrix with entries in $(S, +, \times, 0, 1)$

Definition: The *KLAY-nee* Star

$$A^* = \underbrace{I + A + A^2 + A^3}_{\text{The 3-Cumulant}} + \ldots = (I-A)^{-1}$$

Definition: The k-Cumulant

$$C_k = I + A + A^2 + \ldots + A^k = (I+A)^k$$

(under technical assumptions)

Theorem: (for semi-definite networking problems)

$$\boxed{A^* = C_k = M^k \text{ for } k > n-1}$$

where M=I+A

Bool (existence)

$S = \{\top, \bot\}$

$+ =$ OR and $\times =$ AND

$0 = \bot$ and $1 = \top$

Name (routing)

$S = \{[ij]$ edges$\}$

$+ =$ disjoint union

$\times =$ concatenation

$0 = \emptyset$ and $1 = \sum [i]$

Min-Plus (shortest path)

$S = [0,\infty]$

$+ =$ min and $\times =$ plus

$0 = \infty$ and $1 = 0$

Lifetime

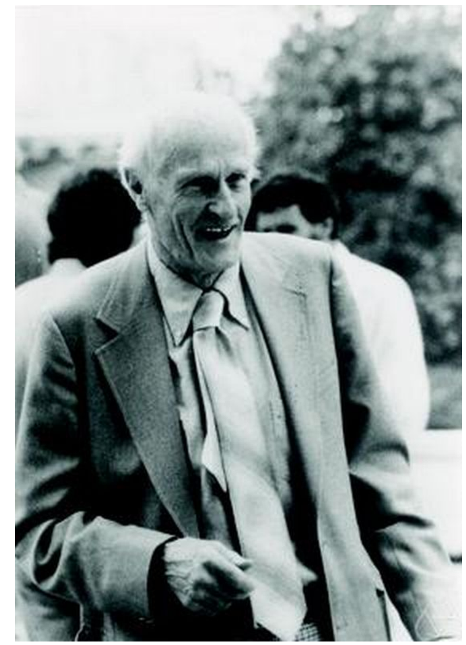
$S = \mathcal{P}(\mathbb{R})$

$+ = \cup$ and $\times = \cap$

$0 = \emptyset$ and $1 = \mathbb{R}$

Our contribution to NASA & HDTN!

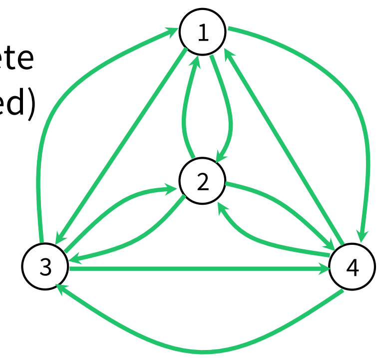
Stephen Kleene (1909-1994)



Strongly Connected Graphs via Kleene Star

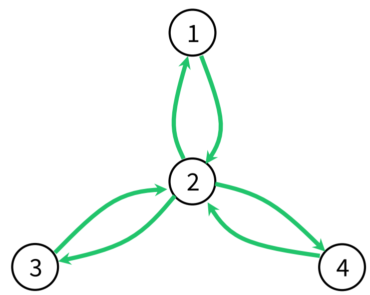
Strongly Connected: Can get from any vertex to any other vertex

Complete (Directed) Graph



$$A = \begin{bmatrix} \bot & \top & \top & \top \\ \top & \bot & \top & \top \\ \top & \top & \bot & \top \\ \top & \top & \top & \bot \end{bmatrix}$$

$A^* = C_1$
i.e. converges after 1 step

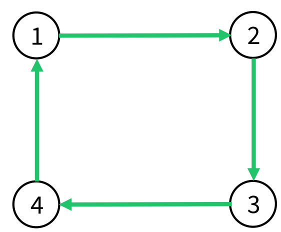


$$A = \begin{bmatrix} \bot & \top & \bot & \bot \\ \top & \bot & \top & \top \\ \bot & \top & \bot & \bot \\ \bot & \top & \bot & \bot \end{bmatrix}$$

$A^* = C_2$
i.e. converges after 2 steps

A directed graph is strongly connected $\Longleftrightarrow$ $A^* = \begin{bmatrix} \top & \top & \top & \top \\ \top & \top & \top & \top \\ \top & \top & \top & \top \\ \top & \top & \top & \top \end{bmatrix}$

$$\boxed{A^* = C_k = M^k \text{ for } k \geq \mathrm{diam}(G)}$$

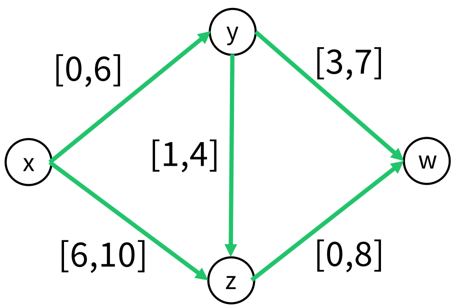


$$A = \begin{bmatrix} \bot & \top & \bot & \bot \\ \bot & \bot & \top & \bot \\ \bot & \bot & \bot & \top \\ \top & \bot & \bot & \bot \end{bmatrix}$$

$A^* = C_3$ i.e. converges after 3 steps

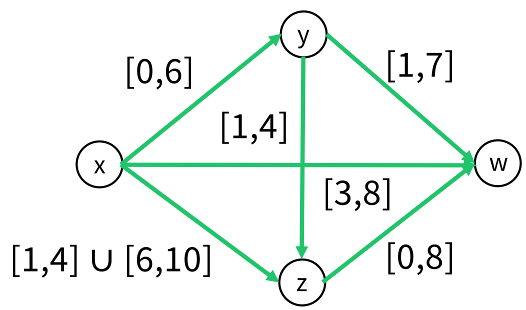
Time-Varying Graphs are Matrices of Lifetimes

TVG

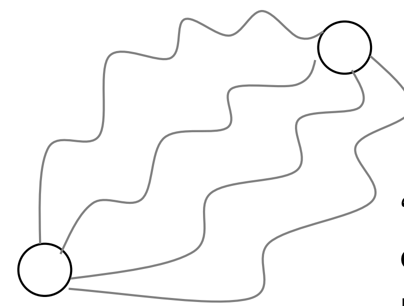


$$M = I + A = \begin{bmatrix} \mathbb{R} & [0,6] & [6,10] & \emptyset \\ \emptyset & \mathbb{R} & [1,4] & [3,7] \\ \emptyset & \emptyset & \mathbb{R} & [0,8] \\ \emptyset & \emptyset & \emptyset & \mathbb{R} \end{bmatrix} \begin{bmatrix} \mathbb{R} & [0,6] & [6,10] & \emptyset \\ \emptyset & \mathbb{R} & [1,4] & [3,7] \\ \emptyset & \emptyset & \mathbb{R} & [0,8] \\ \emptyset & \emptyset & \emptyset & \mathbb{R} \end{bmatrix}$$

$$M^2 = \begin{bmatrix} \mathbb{R} & [0,6] & [1,4] \cup [6,10] & [3,6] \cup [6,8] \\ \emptyset & \mathbb{R} & [1,4] & [1,4] \cup [3,7] \\ \emptyset & \emptyset & \mathbb{R} & [0,8] \\ \emptyset & \emptyset & \emptyset & \mathbb{R} \end{bmatrix}$$

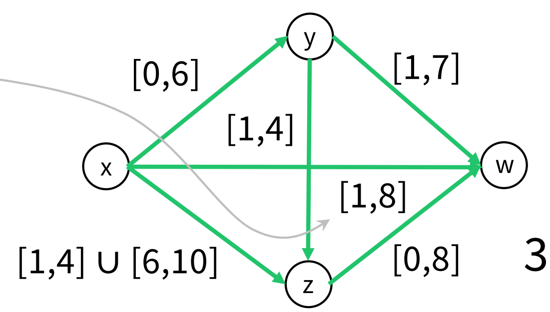


2-Cumulant TVG



“Sum over paths”

$$M^3 = \begin{bmatrix} \mathbb{R} & [0,6] & [1,4] \cup [6,10] & [3,8] \cup [1,4] \\ \emptyset & \mathbb{R} & [1,4] & [1,7] \\ \emptyset & \emptyset & \mathbb{R} & [0,8] \\ \emptyset & \emptyset & \emptyset & \mathbb{R} \end{bmatrix}$$

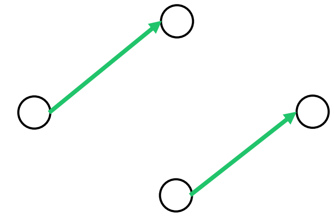


3-Cumulant TVG

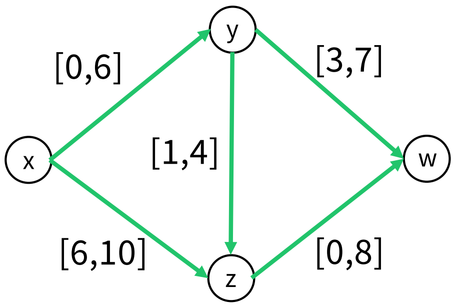
Lifetime Growth : $\emptyset \subseteq [3,8] \subseteq [1,8]$

TVG Vector Multiplication

graph slice at t=0

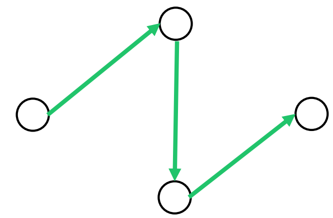


TVG



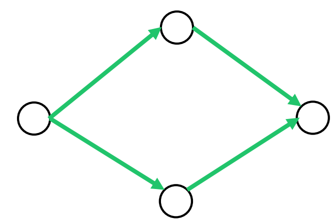
$$[\{0\}\ \ \emptyset\ \ \emptyset\ \ \emptyset] \begin{bmatrix} \mathbb{R} & [0,6] & [6,10] & \emptyset \\ \emptyset & \mathbb{R} & [1,4] & [3,7] \\ \emptyset & \emptyset & \mathbb{R} & [0,8] \\ \emptyset & \emptyset & \emptyset & \mathbb{R} \end{bmatrix} \Rightarrow [\{0\}\ \ \{0\}\ \ \emptyset\ \ \emptyset]$$

graph slice at t=1



$$[\{1\}\ \ \emptyset\ \ \emptyset\ \ \emptyset] \begin{bmatrix} \mathbb{R} & [0,6] & [6,10] & \emptyset \\ \emptyset & \mathbb{R} & [1,4] & [3,7] \\ \emptyset & \emptyset & \mathbb{R} & [0,8] \\ \emptyset & \emptyset & \emptyset & \mathbb{R} \end{bmatrix} \Rightarrow [\{1\}\ \ \{1\}\ \ \emptyset\ \ \emptyset] \begin{bmatrix} \mathbb{R} & [0,6] & [6,10] & \emptyset \\ \emptyset & \mathbb{R} & [1,4] & [3,7] \\ \emptyset & \emptyset & \mathbb{R} & [0,8] \\ \emptyset & \emptyset & \emptyset & \mathbb{R} \end{bmatrix} \Rightarrow [\{1\}\ \ \{1\}\ \ \{1\}\ \ \emptyset]$$

graph slice at t=6



$$[\{6\}\ \ \emptyset\ \ \emptyset\ \ \emptyset] \begin{bmatrix} \mathbb{R} & [0,6] & [6,10] & \emptyset \\ \emptyset & \mathbb{R} & [1,4] & [3,7] \\ \emptyset & \emptyset & \mathbb{R} & [0,8] \\ \emptyset & \emptyset & \emptyset & \mathbb{R} \end{bmatrix} \Rightarrow [\{6\}\ \ \{6\}\ \ \{6\}\ \ \emptyset] \begin{bmatrix} \mathbb{R} & [0,6] & [6,10] & \emptyset \\ \emptyset & \mathbb{R} & [1,4] & [3,7] \\ \emptyset & \emptyset & \mathbb{R} & [0,8] \\ \emptyset & \emptyset & \emptyset & \mathbb{R} \end{bmatrix} \Rightarrow [\{6\}\ \ \{6\}\ \ \{6\}\ \ \{6\}]$$

TVG Matrix Operations

KLEENE-STAR MODELS TIME AVAILABLE INSTANTANEOUS WALKS

VECTOR MULTIPLICATION MODELS TEMPORAL FOOTPRINT OF BROADCAST

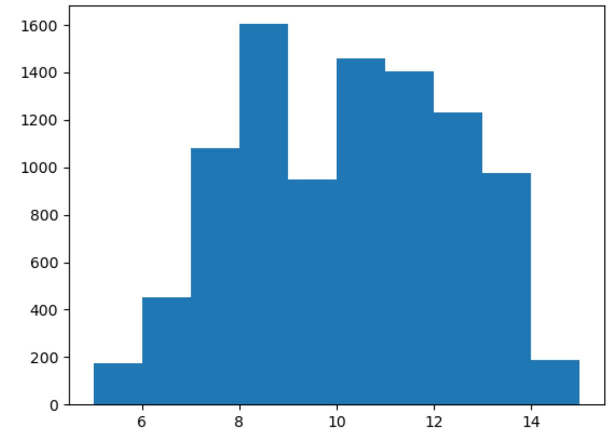
Theorem: (Robby Green, as part of TIMAEUS)

The Kleene Star $A^* = C_k$ for k = temporal diameter, i.e., the largest diameter per component over time.

Proof follows from an isomorphism:

$$\boxed{\mathrm{Mat}_{n \times n}(\mathcal{P}(\mathbb{R})) \cong \mathrm{Fun}(\mathbb{R}, \mathrm{Mat}_{n \times n}(\mathrm{Bool}))}$$

80 Starlink Satellites over One Day



Diameter of Graph Slice

Upshot: Temporal Diameter << number of nodes

Testing proximity to Complete TVG requires lower degree cumulants!

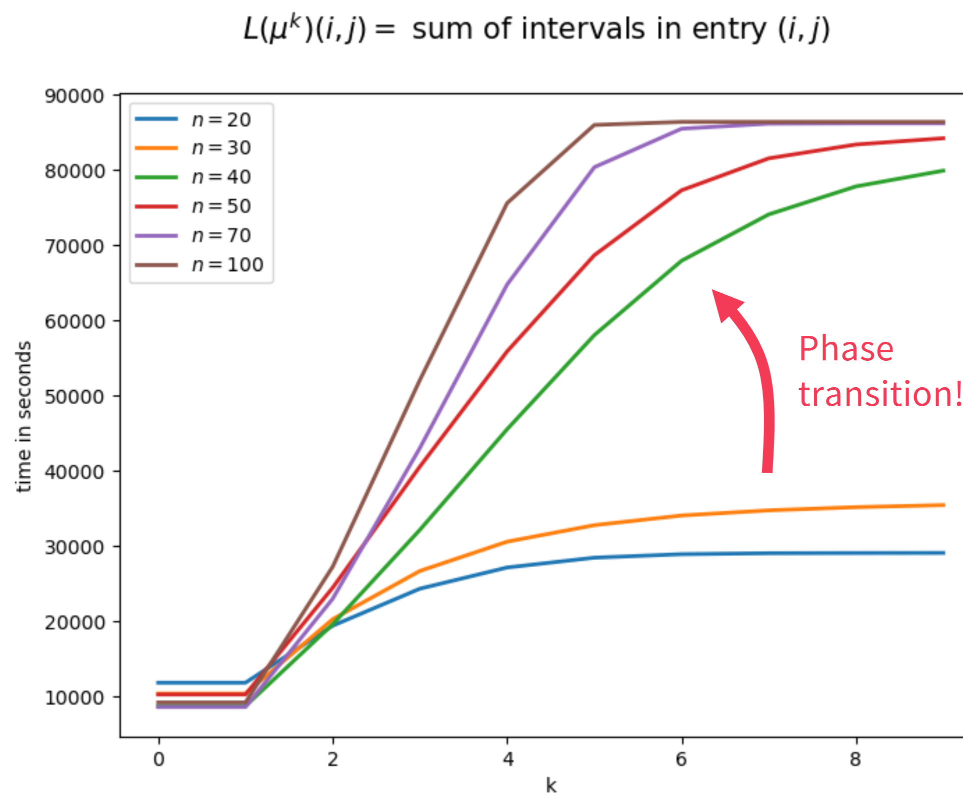
$$A(K_4) = \begin{bmatrix} \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \end{bmatrix}$$

STARLINK
Lifetime Growth

Using Portion Library in Python we can automatically extract TVG matrices from SOAP simulations

M^k models lifetimes of length $\leq$k walks from node to node

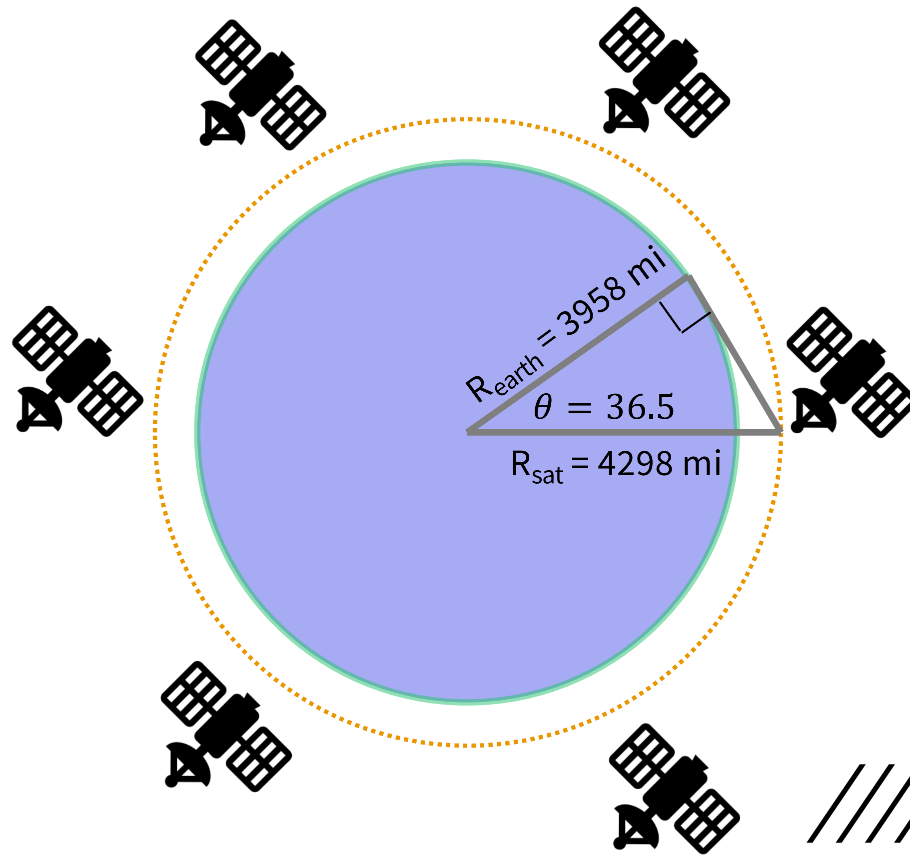
Average Lifetime out of 86,400 seconds models connectivity over 1 day



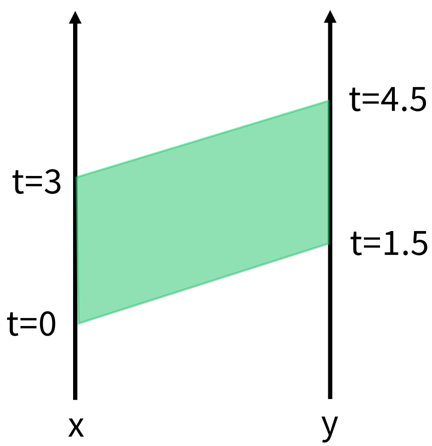
If we randomly sample STARLINK, how close is a sample to a "static" graph; when using longer walks?

Connectivity Properties for STARLINK

- 20 or 30 nodes insufficient for strong connectivity
- Phase transition around n=40 for strong connectivity
- Convergence radius tapers from **k=9** (for n=50) to **k=7** (for n=70) to **k=5** (for n=100)
- With ~340 mile altitude, **k=3** is the *conjectured* limiting behavior



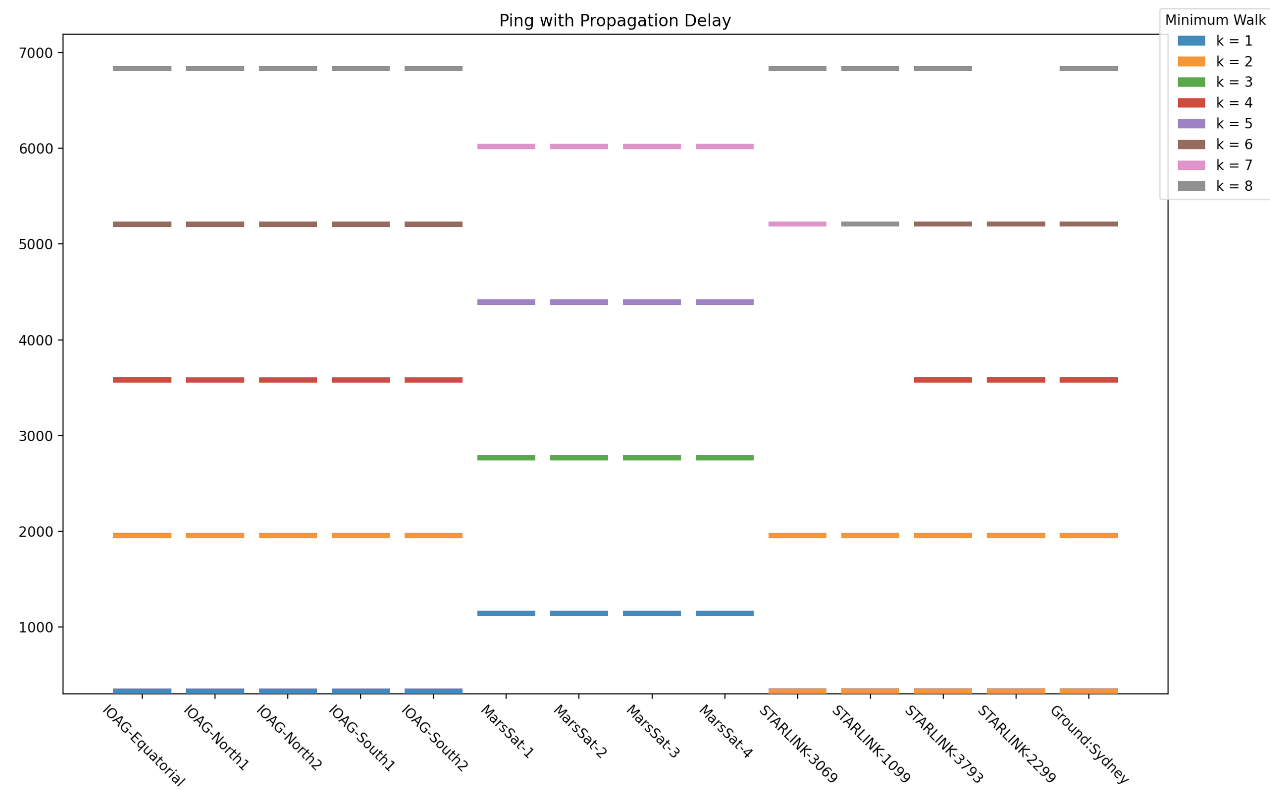
Modeling Propagation Delay (Using another semiring)



$$A = \begin{bmatrix} \emptyset & ([0,3], 1.5) \\ \emptyset & \emptyset \end{bmatrix}$$

Interpret this as element in $\mathrm{End}(\mathcal{P}(\mathbb{R}))$

- intersect with [0,3]
- shift by 1.5 units to the right
- $\oplus$ is union of images
- $\otimes$ is composition of operators



Kleene Star doesn’t converge = Difficulty of Deep Space Routing!

50 Years of Algebraic Network Theory

J. Inst. Maths Applics (1971) **7**, 273–294

An Algebra for Network Routing Problems

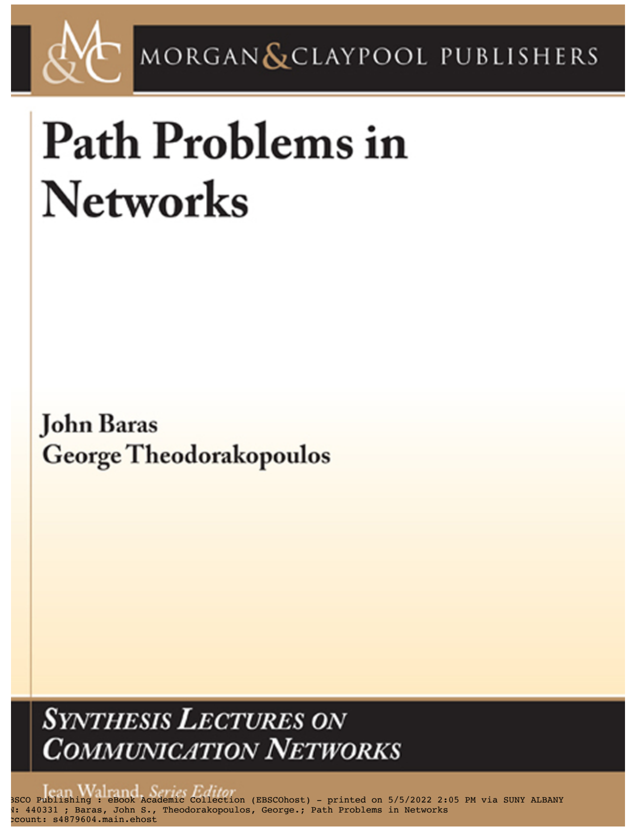
B. A. Carré

Department of Electrical Engineering, University of Southampton, England

[Received 15 May 1970 and in revised form 25 June 1970]

Problems involving the determination of routes on networks arise in many different contexts. For example network flow problems in operations research, such as transportation and assignment problems, involve the determination of a succession of shortest or least-cost paths between commodity sources and sinks. Again, critical path analysis and certain scheduling problems involve the determination of longest paths on activity networks. Pathfinding problems of different kinds also arise in the design of logic networks, and in routing messages through congested communication networks. This paper presents an algebraic structure for the formulation and solution of such problems.

After defining the algebraic structure and giving concrete examples applicable to different kinds of routing problems, we use it in a general analysis of a class of directed networks, in which each arc has an associated measure (representing for instance a transportation cost, an activity duration, the state (open or closed) of a switch, or the probability of a communication link being available). It is then shown that all the routing problems mentioned above can be expressed in the same algebraic form, and that they can all be solved by variants of classical methods of linear algebra, differing from these only in the significance of the additive and multiplicative operations.



Application	S	$\oplus$	$\otimes$
Path enumeration	Subsets of $\cup_{i=1}^{n} V^i$	Set union	Latin multiplication
Markov chain stationary distribution	$[0, \infty) \cup \{\infty\}$	$+$	$\times$
Expected number of visits of random walk from vertex s to t	$[0, \infty) \cup \{\infty\}$	$+$	$\times$
Expected cost of paths from vertex s to t	$\{(p, c) \in ([0, \infty) \times \mathbb{R})\}$	$(p_1 + p_2, c_1 + c_2)$	$(p_1 p_2, p_1 c_2 + p_2 c_1)$
Minimum weight spanning tree	$\mathbb{R}$	min	max
Longest path	$\mathbb{R} \cup \{-\infty\}$	max	$+$
Shortest path	$\mathbb{R} \cup \{\infty\}$	min	$+$
Widest path	$[0, \infty) \cup \{\infty\}$	max	min
Most reliable path	$[0, 1]$	max	$\times$
Widest-shortest path	$\{(d, b) \in [0, \infty) \times [0, \infty)\}$	lexicographic min	$(d_1 + d_2, \min(b_1, b_2))$
k-shortest paths	$a, b \in \{\mathbb{R} \cup \{\infty\}\}^k$	min-k$\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$	min-k$\{a_i + b_j \mid i, j = 1, \ldots, k\}$
BGP routing	$\{c, p, r, ①, ⓪\}$	see Table 4.1	see Table 4.1
Shortest path with time-inhomogeneous edges	$w_1, w_2 : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ and $t \to \infty \Rightarrow w(t) \nearrow \infty$	$\min\{w_1(t), w_2(t)\}$	$w_2(w_1(t))$
Reachability under deterministic edge failures	$\{0, 1\}$	max	min
Bridge and cut vertices	$2^E \cup \{⓪\}$	Set intersection	Set union
Reachability under probabilistic edge failures	formal polynomials w_e in the variables $x_{e_i}, e \in E, i \in \mathbb{N}$	$w_1 + w_2 - w_1 \otimes w_2$	$\prod\{\text{all } x_e \text{ in } w_1 \text{ or } w_2\}$
Shortest paths with edge gains or losses	$\{(c, \mu) \in \mathbb{R} \times (0, \infty)\}$	$\begin{cases} (c_1, \mu_1) & \text{if } \frac{c_1}{\mu_1} < \frac{c_2}{\mu_2} \\ (c_1, \mu_1) & \text{if } \frac{c_1}{\mu_1} = \frac{c_2}{\mu_2} \text{ and } \mu_1 > \mu_2 \\ (c_2, \mu_2) & \text{otherwise} \end{cases}$	$(c_1 + \mu_1 c_2, \mu_1 \mu_2)$

From Baras and Theodorakopoulos “Path Problems in Networks” 2010

Application	S	$\oplus$	$\otimes$
Trust: Path semiring	$\{(t, c) \in [0,1] \times [0,1]\}$	$(t_1 t_2, c_1 c_2)$	$\begin{cases} (t_1, c_1) & \text{if } c_1 > c_2 \\ (t_2, c_2) & \text{if } c_1 < c_2 \\ (\max\{t_1, t_2\}, c_1) & \text{if } c_1 = c_2 \end{cases}$
Trust: Distance semiring	$\{(a, b) \in ([0, \infty] \times [0,1])\}$	$(a_1 + a_2, b_1 + b_2)$	$(a_1 a_2, a_1 b_2 + a_2 b_1)$
Trust: PGP computation model	$a_1, a_2 \in \mathbb{N}^k, k \in \mathbb{N}$	$+$	$\mathtt{valid}(a_1) a_2$
Trust: EigenTrust	$[0, 1]$	$+$	$\times$
Trust: Semantic web	$[0, 1]$	max	$\times$
Social: balance semiring	$\{0, -1, 1, a\}$	see Table 4.2	see Table 4.2
Social: cluster semiring	$\{0, -1, 1, a, b\}$	see Table 4.3	see Table 4.3
Social: Geodetic semiring (length and number of shortest paths)	$\{(d, n) \in ([0, \infty) \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})\}$	$\begin{cases} (d_1, n_1) & \text{if } d_1 < d_2 \\ (d_2, n_2) & \text{if } d_1 > d_2 \\ (d_1, n_1 + n_2) & \text{if } d_1 = d_2 \end{cases}$	$(d_1 + d_2, n_1 n_2)$
Social: Geosetic semiring (set of vertices of shortest paths)	finite sets with quadruples of the form $(i, P, j, d), i, j \in V, P \subseteq V, d \in \mathbb{N}$	see Sec. 4.11.4	see Sec. 4.11.4
Traffic assignment	$a_1, a_2 \in [0, \infty)$	$-\frac{1}{\mu} \log \left(e^{-\mu a_1} + e^{-\mu a_2} \right)$	$a_1 + a_2$
Edge sensitivity: Traffic attraction attack	$[0, \infty) \cup \{\infty\}$	min	$+$
Edge sensitivity: Traffic repulsion attack	$[0, \infty) \cup \{\infty\}$	min	$+$

From Baras and Theodorakopoulos “Path Problems in Networks” 2010

OUR TVG DEFINITION IS NEW!

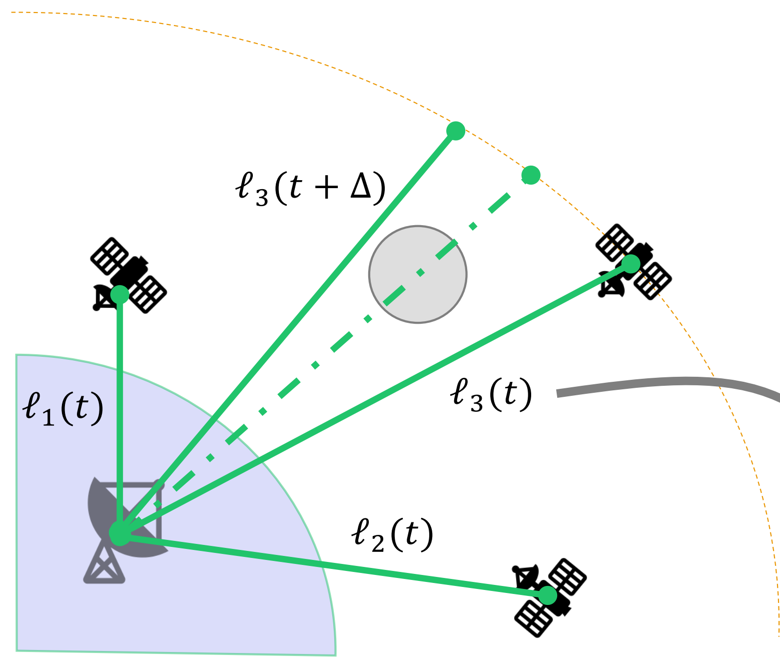
(SURPRISINGLY)

METRICS ON TVGS

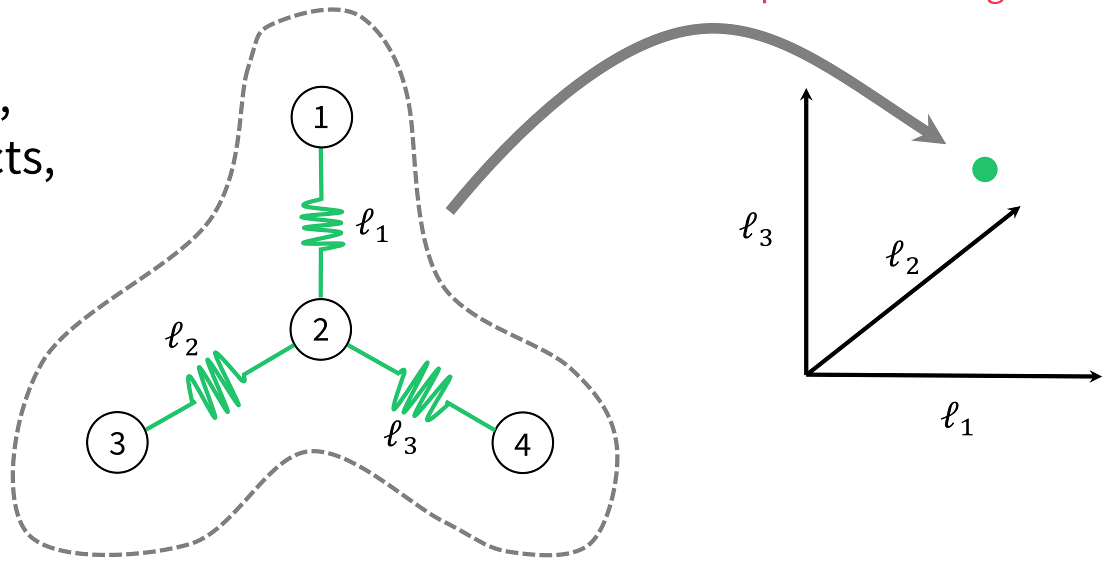
COORDINATIZING THE SPACE OF ALL NETWORKING SCENARIOS

The Difficulty of Continuity

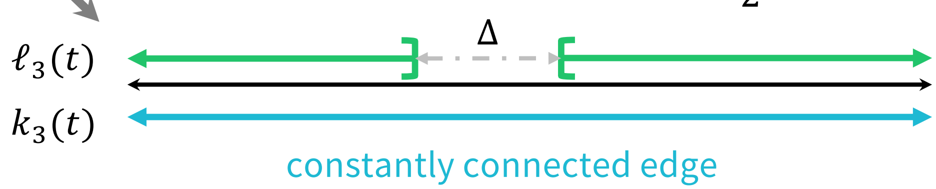
Space networking has both continuous, e.g., signal strength, and discrete aspects, e.g., line of sight.



Traditional methods ***fail*** to capture line of sight

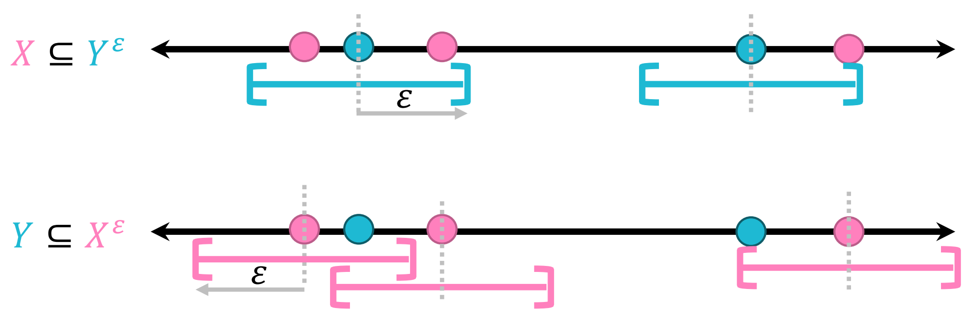


But subsets of $\mathbb{R}$ (TVG Coordinates) have metrics where these are $\frac{\Delta}{2}$ - close!



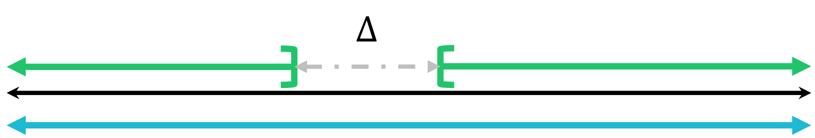
Hausdorff Distance

Two subsets X and Y are ε-close if $X \subseteq Y^{\varepsilon}$ and $Y \subseteq X^{\varepsilon}$



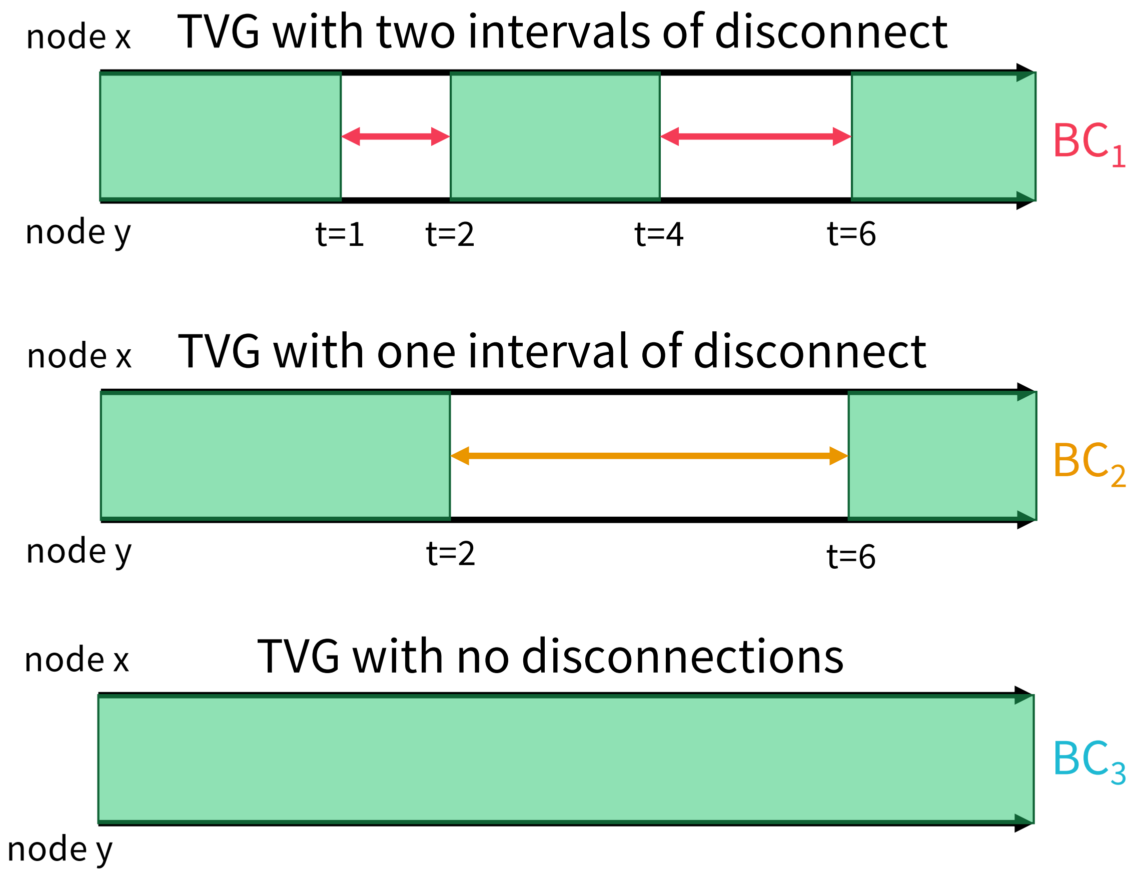
$$d_{\text{Haus}}(X, Y) = \inf\{\varepsilon \mid X \subseteq Y^{\varepsilon} \text{ and } Y \subseteq X^{\varepsilon}\}$$

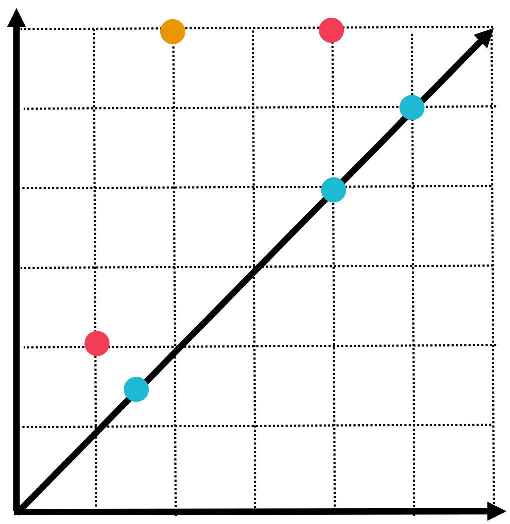
Check our example!



Felix Hausdorff (1868-1942)

Disconnect (Bottleneck) Distance on TVGs





$$\boxed{d_B = \min_{\sigma} \max_{p} \|p - \sigma(p)\|_\infty}$$

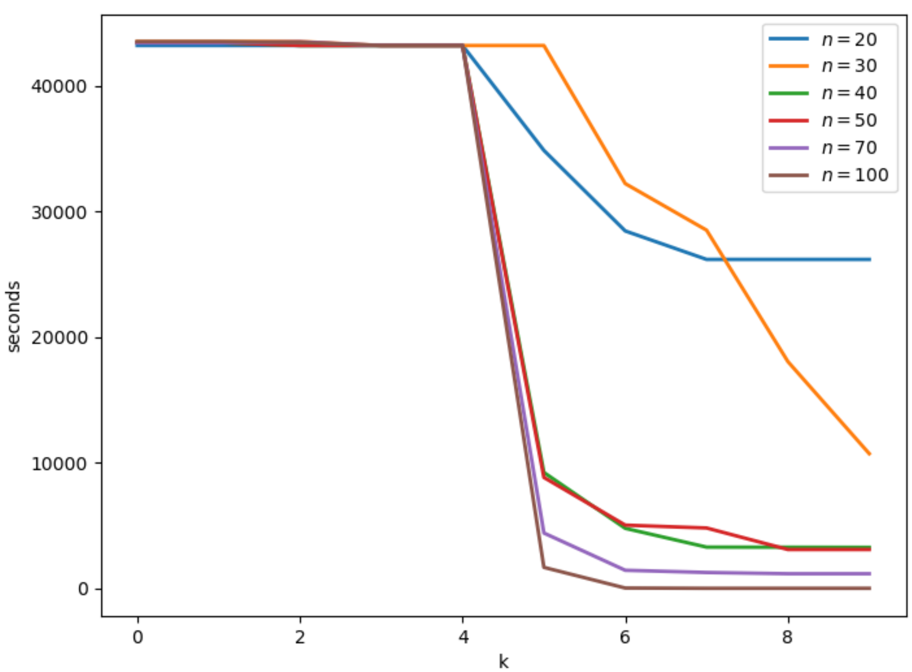
Effectively Computable
via Hungarian Algorithm

STARLINK Distance Curves

measures distance to (bigger version of) this!

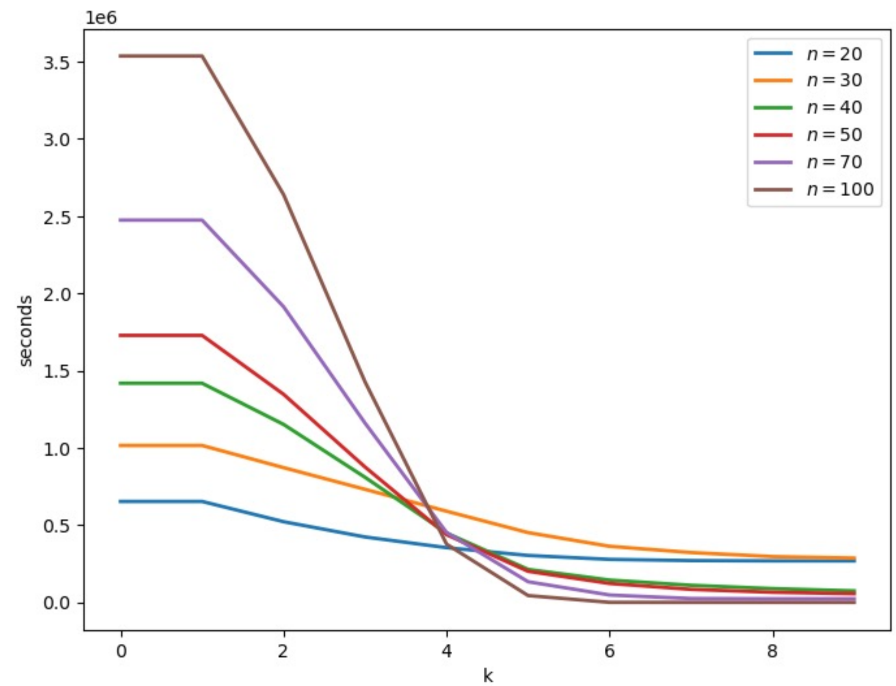
$$\begin{bmatrix} \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \end{bmatrix}$$

Bottleneck Distance from *k*-Walk Matrix to Complete Matrix



$$d_B = \min_{\sigma} \max_{p} \|p - \sigma(p)\|_\infty$$

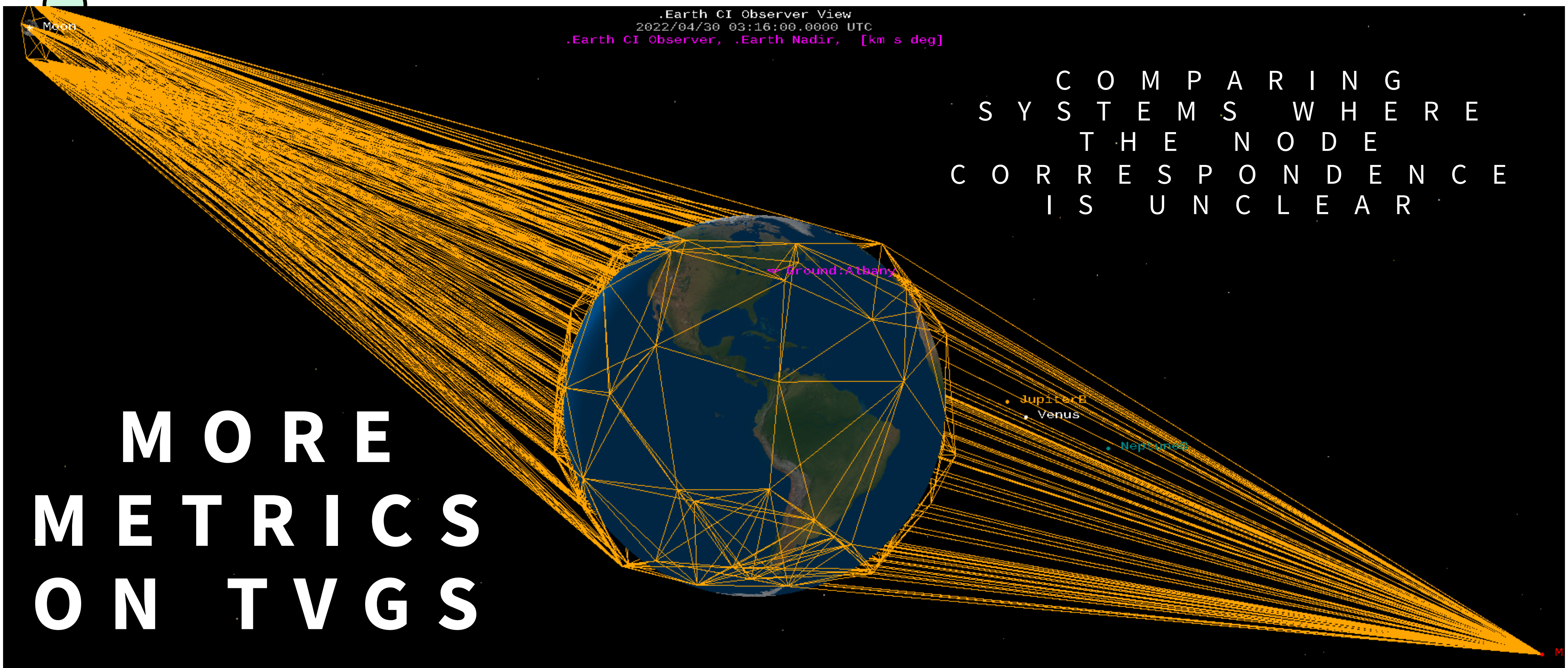
2-Wasserstein Distance from *k*-Walk Matrix to Complete Matrix



$$d_{\text{2-Wasserstein}} = \min_{\sigma} \sqrt[2]{\textstyle\sum_p \|p - \sigma(p)\|_2}$$

MORE METRICS ON TVGS

COMPARING SYSTEMS WHERE THE NODE CORRESPONDENCE IS UNCLEAR

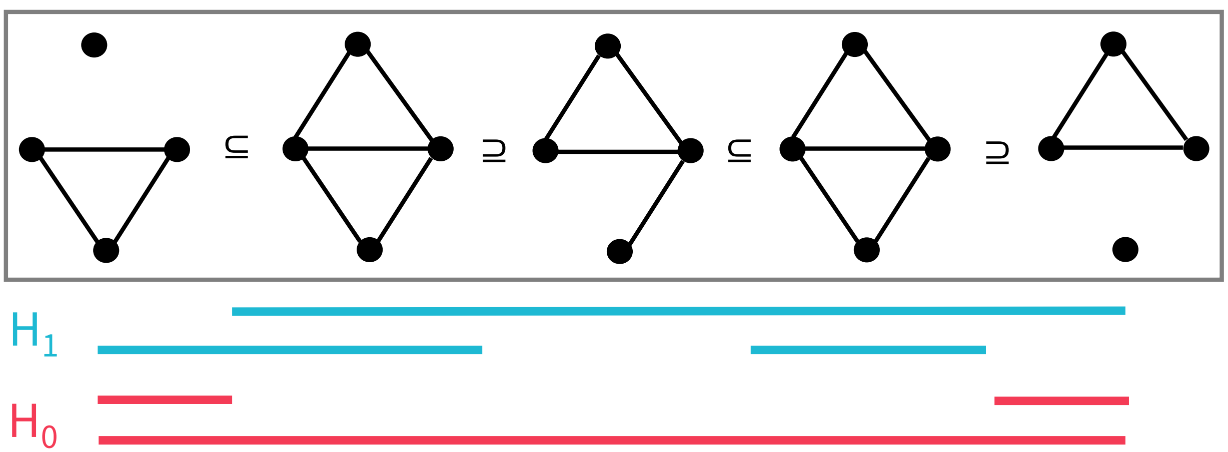


Dimensionality Reduction via Topological Featurization

Computational Complexity Concerns

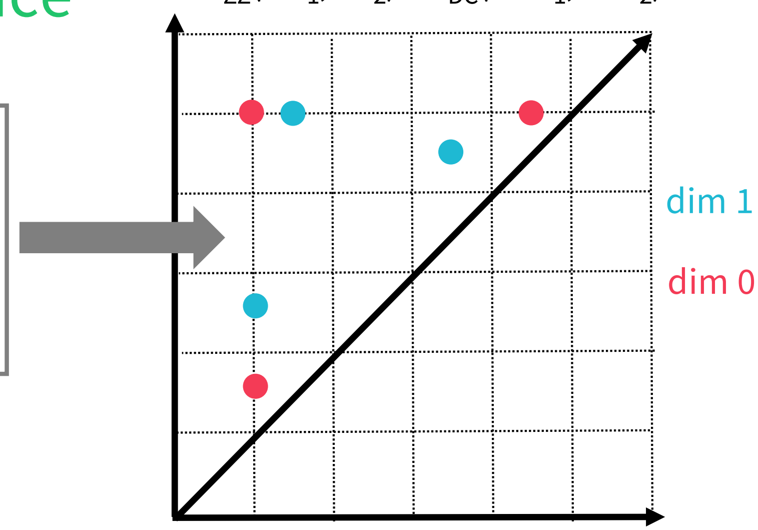
- Best-case for Disconnect Distance: $O(n^2 \times n^3)$
- Worst-case for Disconnect Distance: $O(n! \times n^2 \times n^3)$
- Topological Reduction reduces to: $O(n^3)$

Zig-Zag Persistence



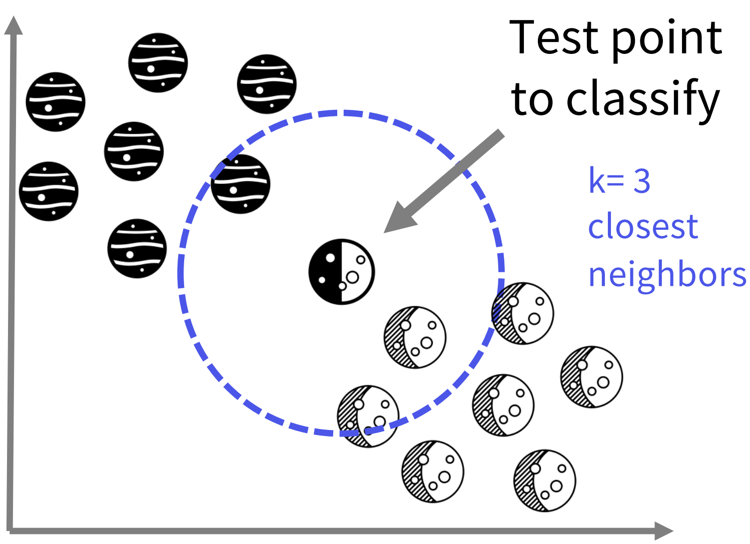
THM: STABILITY

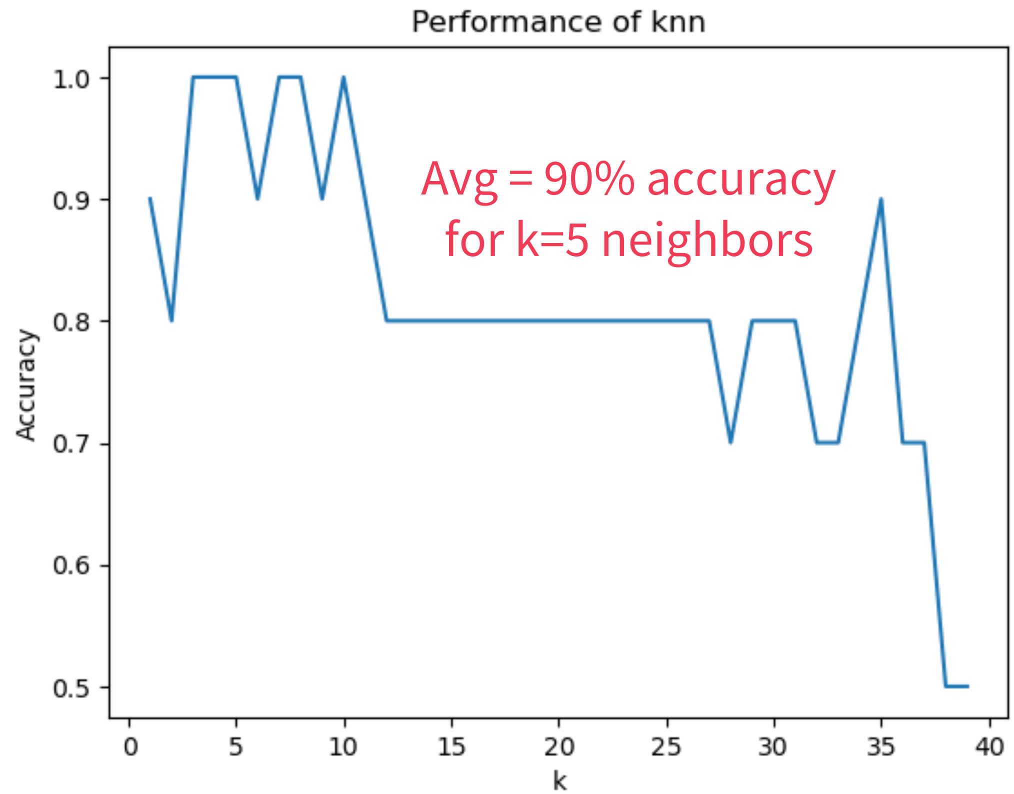
$d_{ZZ}(BC_1,BC_2) \leq d_{DC}(TVG_1,TVG_2)$



Earth + Mars vs. Earth + Moon Network Comparison

Took 24 samples from 2 classes:
- 15 Starlink + 5 Moon Sats
- 15 Starlink + 5 Mars Sats



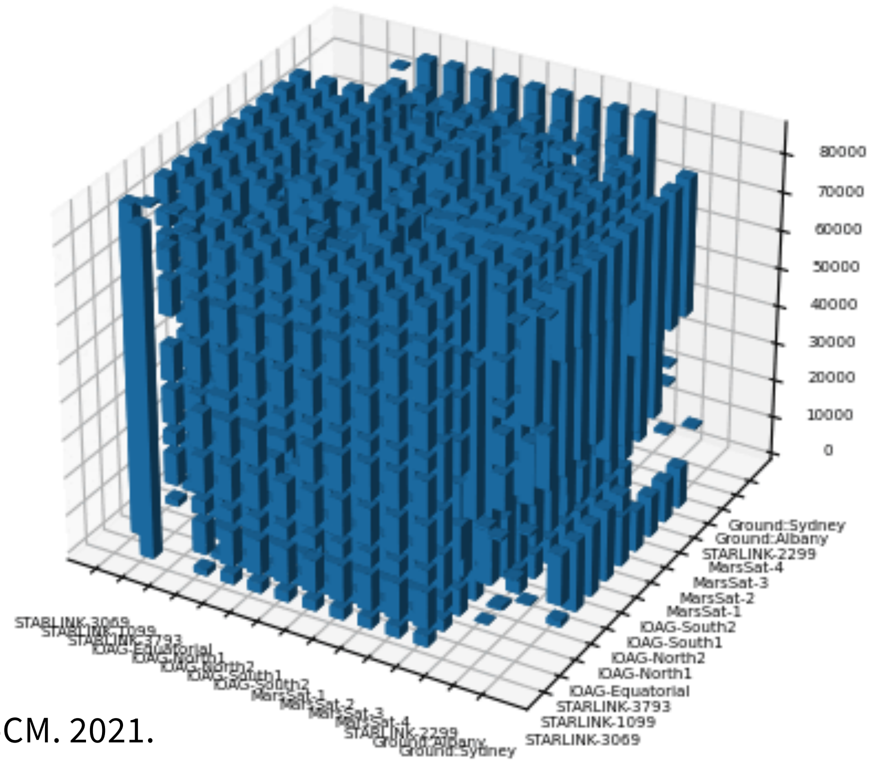


K Nearest Neighbors Classifier

Distance measured in zigzag diagram space using H_1

Discussion of Distance Calculations

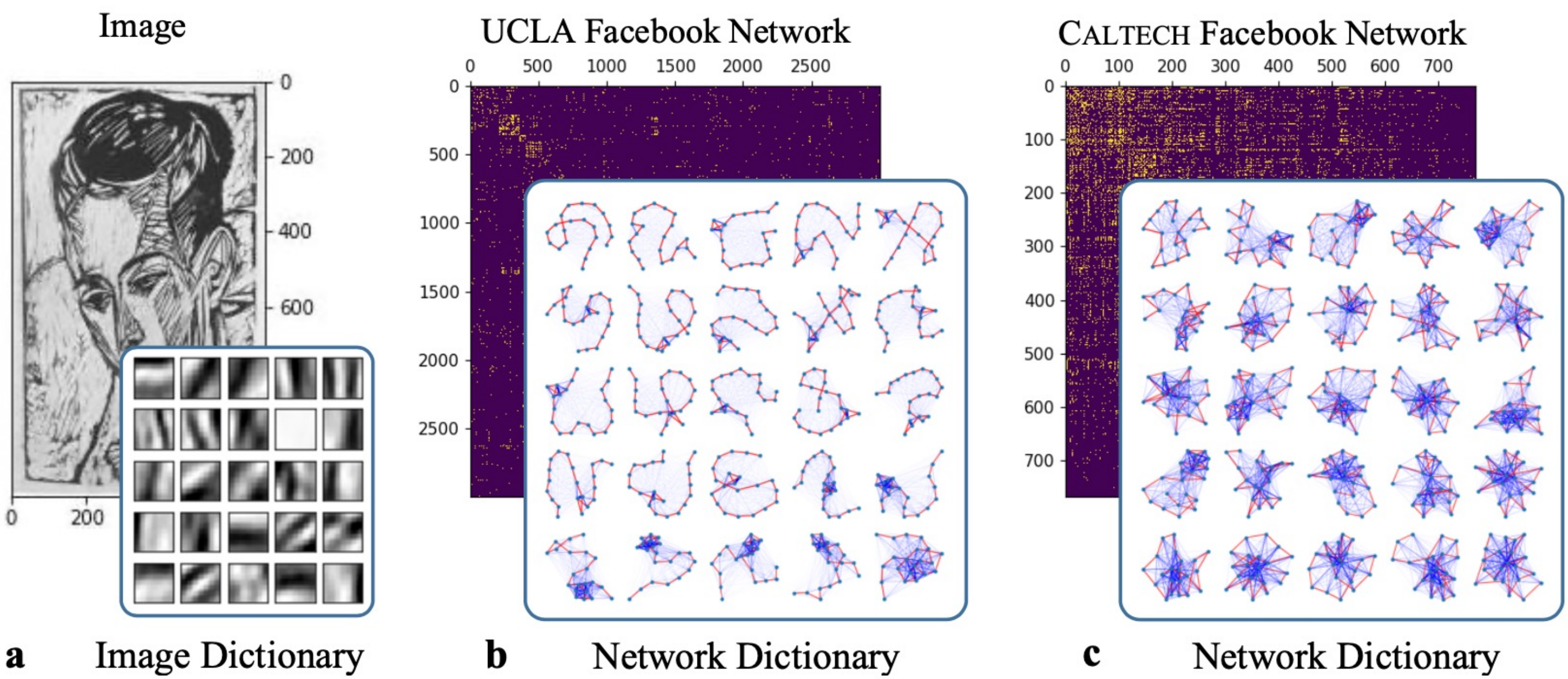
1. Quantify how close to a strongly connected TVG system
2. Necessary for control applications---steer a system from a disconnected one to a connected one!
3. First steps in identifying clusters ("motifs") in TVG space.



[2]: "A Framework for Differential Calculus on Persistence Barcodes" Leygonie, Tillman, and Oudot. FoCM. 2021.

[3]: "Learning Low-Rank Latent Mesoscale Structures in Networks" by Lyu, Kureh, Vendrow, and Porter. arXiV:2102.06984

Motifs for Dictionary Learning



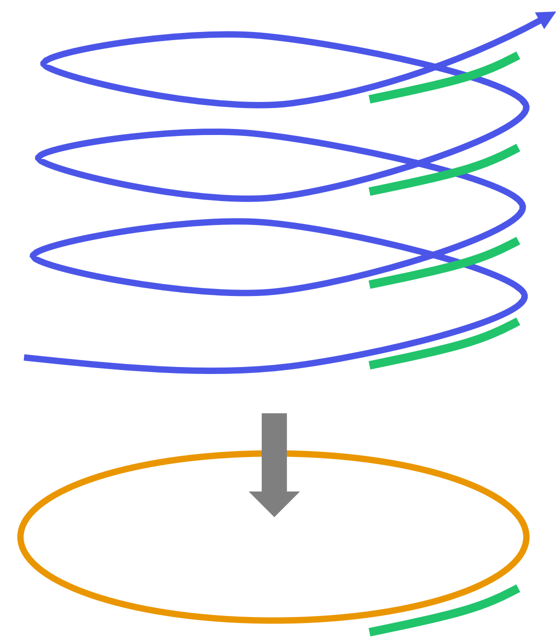
From "Learning Low-Rank Latent Mesoscale Structures in Networks" by Lyu, Kureh, Vendrow, Porter

FUTURE DIRECTIONS

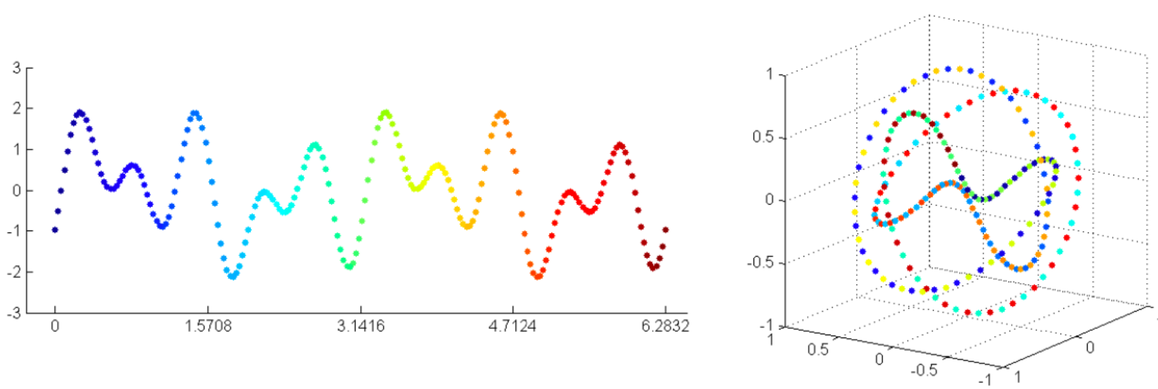
GLUING TOGETHER ROUTING PROTOCOLS USING SHEAVES

Leveraging Periodicity

- Convergence of Kleene-star fails for propagation delay semiring b/c $w(\gamma) \not\preccurlyeq 1 \Leftrightarrow$ Non-trivial shift
- Learn periodicity of TVG subnets
- Replace $\mathcal{P}(\mathbb{R})$ with $\mathcal{P}(\mathbb{S}^1)$ or $\mathcal{P}(\mathbb{T}^n)$
- Study clustergrams over $\mathbb{S}^1$ or $\mathbb{T}^n$



single representative for periodic contact windows

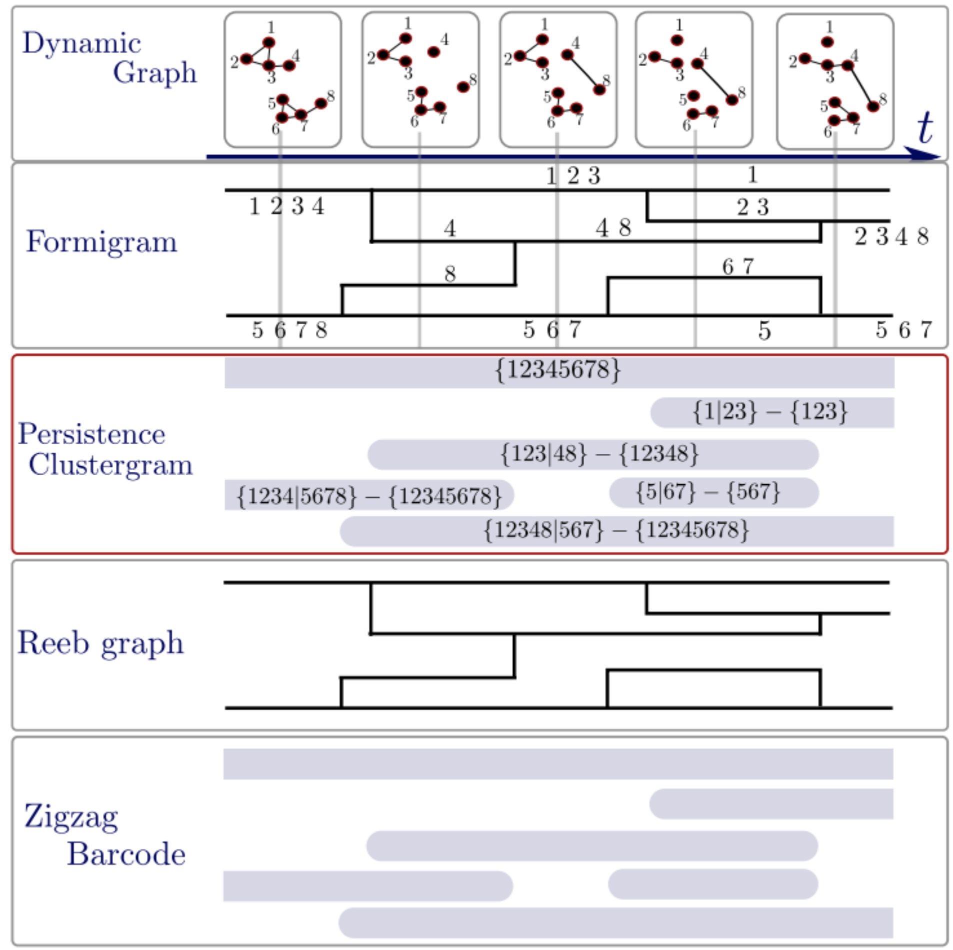


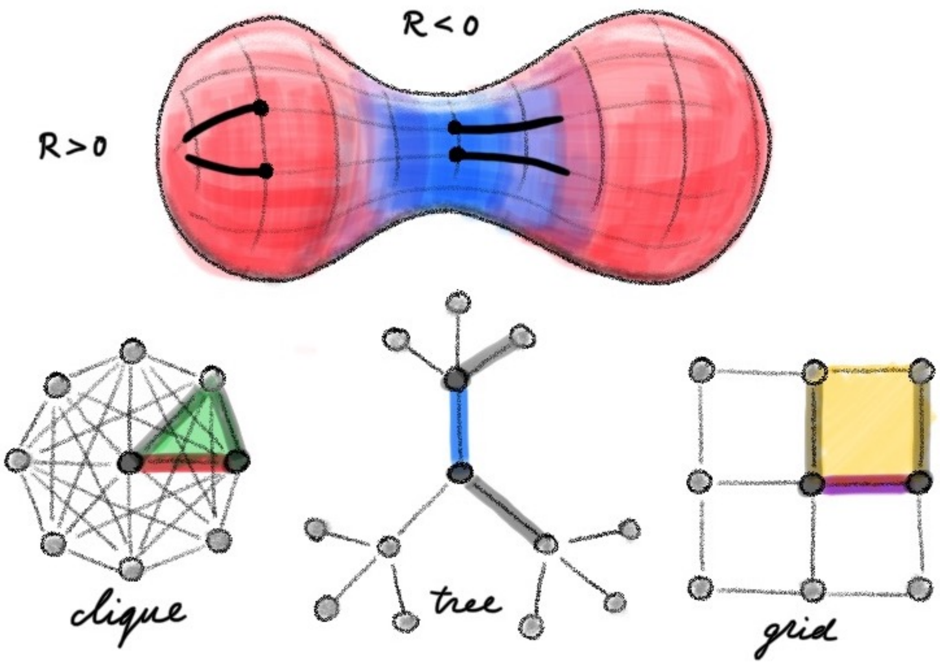
From “Sliding Windows and Persistence” by Perea and Harer. FoCM 2015.

Sub-Netting Methods for TVGs

w/ Woojin Kim (Duke)

Persistence-Based



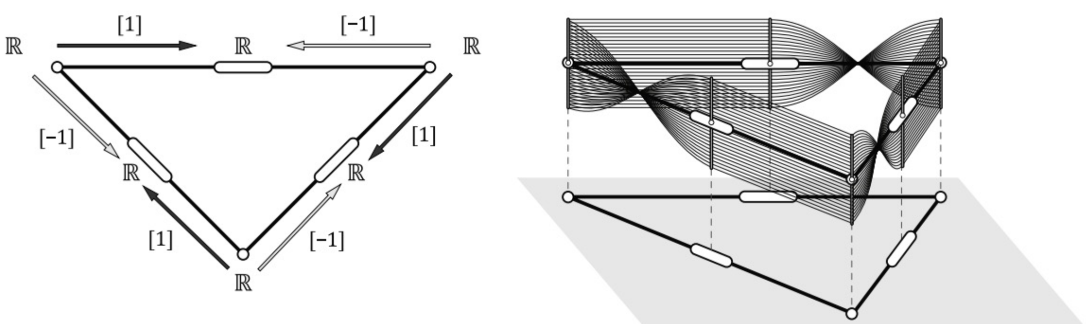


Temporal Curvature-Based

w/ Brendan Mallery (Tufts) + Alan Hylton (Goddard)

Learning Sheaves for Routing

- **“Small” Routing sheaves:** assign to a fixed graph a routing procedure
- **“Big” TVG Routing Sheaves:** assigns to every possible TVG a routing protocol
- **Goal:** *Learn optimal assignments* via deep learning and sheaf Laplacians



“Cellular Sheaves of Lattices and the Tarski Laplacian” by Ghrist and Reiss. HHA 2022.

“Neural Sheaf Diffusion for Deep Learning on Graphs” by Bodnar, Bronstein, and Di Giovanni

THE TIMAEUS PROJECT

Billy Bernardoni, Robert Cardona, Jacob Cleveland, Justin Curry, Jordan deSha
Robby Green, Brian Heller, Alan Hylton, Brendan Mallery, Bob Short
Not Pictured: Tung Lam

THE TIMAEUS PROJECT

Topologically **I**nspired **M**ethods for **A**dhoc **E**volving and **U**ncertain **S**ystems

Simulation Work
R. Cardona, B. Heller

Semirings
B. Bernardoni, J. Curry, R. Green, T. Lam, B. Heller

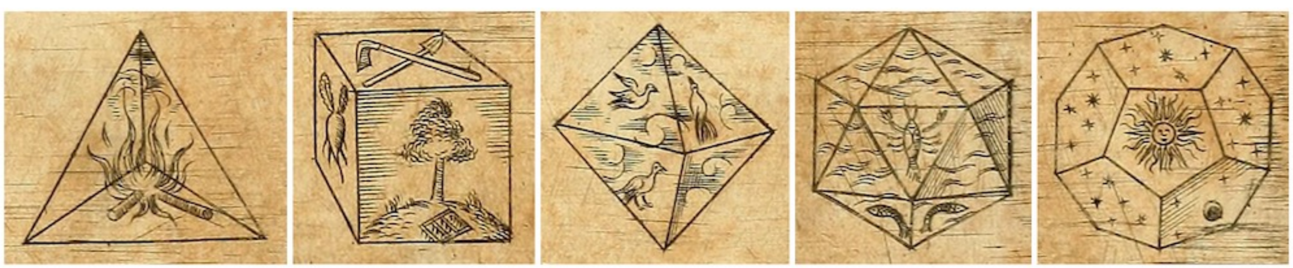
Tropical Geometry
B. Bernardoni, J. Cleveland

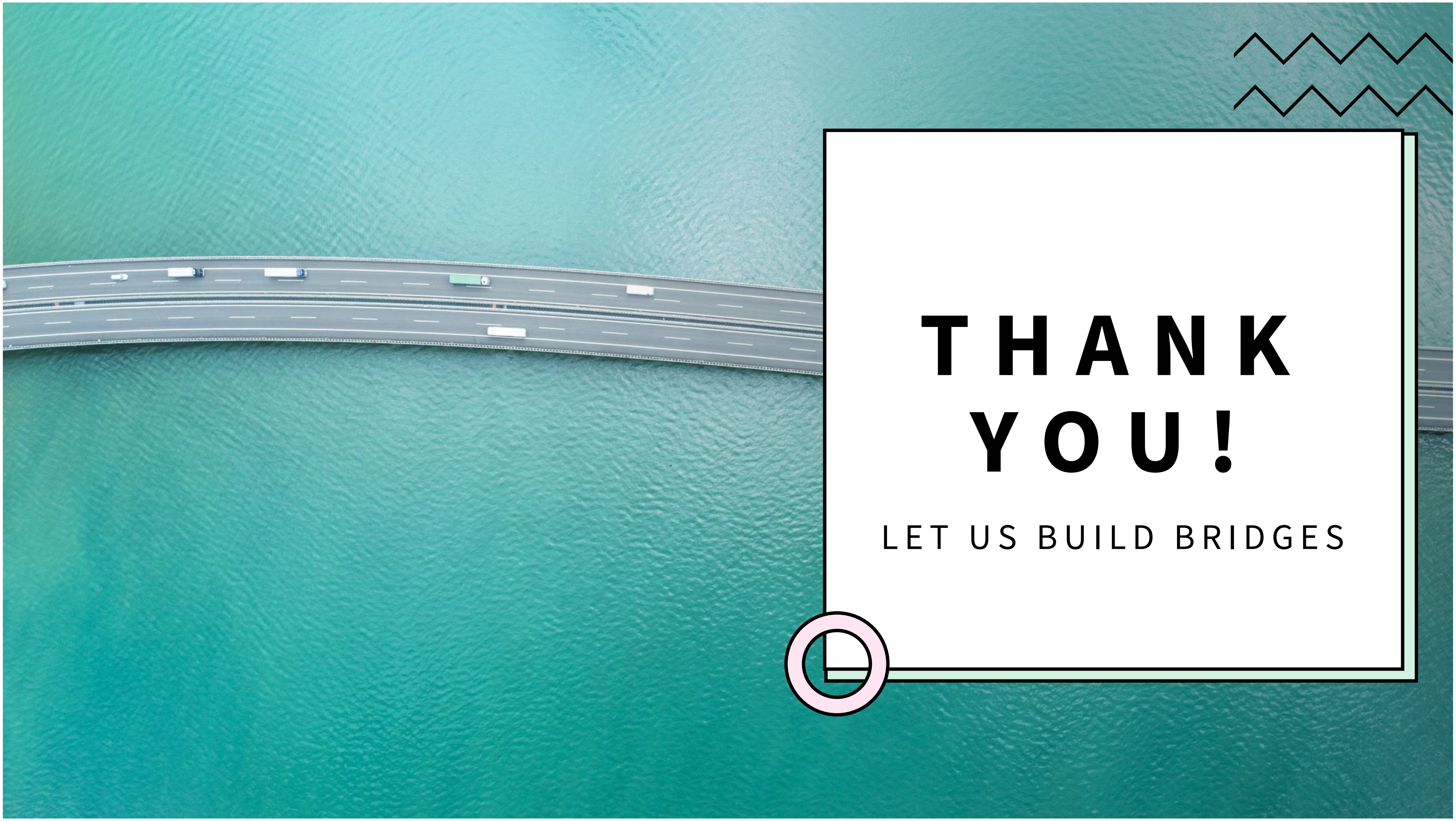
TDA Foundations
R. Cardona, J. Curry, T. Lam

Sheaves and Cosheaves
R. Cardona, J. Curry, A. Hylton, B. Short

Network Curvature
B. Mallery

Hypergraphs
R. Green



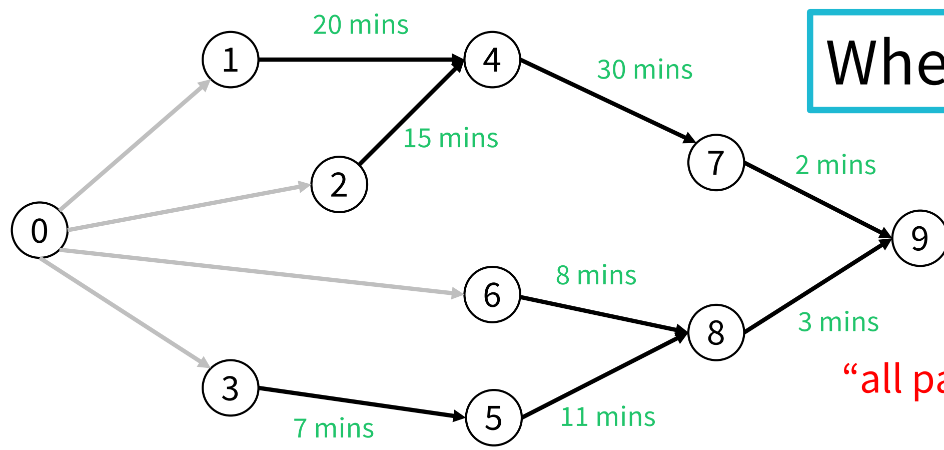

THANK YOU!
LET US BUILD BRIDGES

Critical Path Analysis

(The opposite of earliest time of arrival)

Cooking Dinner

1. Preheat Oven (20 mins)
2. Chop Veggies (15 mins)
3. Boil Water (7 mins)
4. Bake Veggies (30 mins)
5. Cook Pasta (11 mins)
6. Heat Sauce (8 mins)
7. Plate Veggies (2 mins)
8. Plate Pasta (3 mins)
9. Eat dinner!



When's Dinner!?

A^* solves "all pairs longest/shortest path problem"

Semiring Model

$S = \mathbb{R} \cup \{-\infty\}$

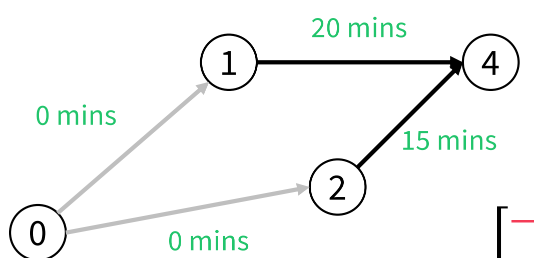
$\oplus = \max$

"0" $= -\infty$

$\otimes = +$

"1" $= 0$

Max-Plus Semiring



If $b = [1700 \quad -\infty \quad -\infty \quad -\infty]$

then $y = [1700 \quad 1700 \quad 1700 \quad 1720]$

$$A = \begin{bmatrix} -\infty & 0 & 0 & -\infty \\ -\infty & -\infty & -\infty & 20 \\ -\infty & -\infty & -\infty & 15 \\ -\infty & -\infty & -\infty & -\infty \end{bmatrix}$$

$$\boxed{\begin{array}{c} y = yA \oplus b \\ \Leftrightarrow \\ y = bA^* \end{array}}$$