Generalized existential completions and applications

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Overview

- Introduce the notion of existential free elements for Lawvere doctrines
- Characterize the generalized existential completion
- Present applications to the Dialectica interpretation

D. Trotta (2020), *The Existential Completion*, Theory and Applications of Categories, 2020, Vol. 35, No. 43.

M.E. Maietti D. Trotta (2023), *A characterization of generalized existential completions*, Annals of Pure and Applied Logic, Vol. 174, No. 4.

D. Trotta, M. Spadetto and V. de Paiva (2023), *Dialectica Principles via Gödel Doctrines*, Theoretical Computer Science, Vol. 947.

D. Trotta, M. Spadetto and V. de Paiva (2021), *The Gödel fibration*, Mathematical Foundations of Computer Science (MFCS).

Hyperdoctrines in categorical logic

- Hyperdoctrines;
- fibrations;
- triposes;
- elementary and existential doctrines;
- ...

F.W. Lawvere (1969), *Adjointness in foundations*, Dialectica

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J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts (1980), *Tripos theory*, Math. Proc. Camb. Phil. Soc.

M.E. Maietti, G. Rosolini (2012), *Quotient completion for the foundation of constructive mathematics*, Log. Univer.

Conjunctive doctrines

Definition

A **conjunctive doctrine** is a functor $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ from the opposite of the category $\mathcal{C}$ to the category of inf-semilattices.

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Example

Let $\mathcal{C}$ be a category with finite limits. The functor

$$\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

assigns to an object A in $\mathcal{C}$ the poset $\mathrm{Sub}_{\mathcal{C}}(A)$ of subobjects of A in $\mathcal{C}$ and, for an arrow $B \xrightarrow{f} A$ the morphism $\mathrm{Sub}_{\mathcal{C}}(f)\colon \mathrm{Sub}_{\mathcal{C}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(B)$ is given by pulling a subobject back along f.

Examples

Example

Let $\mathcal{C}$ be a category with finite limits. The conjunctive doctrine of **weak subobjects (or variations)** is given by the functor

$$\Psi_{\mathcal{C}} \colon \mathcal{C}^{\text{op}} \longrightarrow \textbf{InfSL}$$

where $\Psi_{\mathcal{C}}(A)$ is the poset reflection of the slice category $\mathcal{C}/A$.

Example

Let $\mathcal{A}$ be a locale. The **localic conjunctive doctrine** is given by the functor:

$$\mathcal{A}^{(-)} \colon \text{Set}^{\text{op}} \longrightarrow \textbf{InfSL}$$

assigning $I \mapsto \mathcal{A}^I$. The partial order is provided by the pointwise partial order on functions $f \colon I \longrightarrow \mathcal{A}$.

Syntactic doctrines

Let $\mathcal{L}_{=,\exists}$ be the $(\top, \wedge, =, \exists)$-fragment of first-order Intuitionistic logic (also called regular). We define a conjunctive doctrine:

$$\mathrm{LT}_{=,\exists} \colon \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=,\exists}} \longrightarrow \mathbf{InfSL}$$

where $\mathcal{C}_{\mathcal{L}_{=,\exists}}$ is the category of lists of variables and term substitutions:

- **objects** are finite lists of variables $\vec{x} := (x_1, \dots, x_n)$;
- a **morphism** from $(x_1, \dots, x_n)$ into $(y_1, \dots, y_m)$ is a list $[t_1/y_1, \dots, t_m/y_m]$ where the terms t_i are built in $\mathcal{L}_{=,\exists}$ on the variable $x_1, \dots, x_n$.

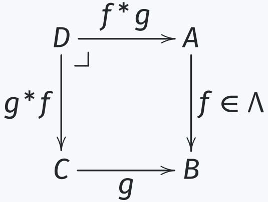
The functor $\mathrm{LT}_{=,\exists} \colon \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=,\exists}} \longrightarrow \mathbf{InfSL}$ sends a list $(x_1, \dots, x_n)$ to the partial order $\mathrm{LT}_{=,\exists}(x_1, \dots, x_n)$ of equivalence classes $[\phi]$ of well formed formulas ϕ in the context $(x_1, \dots, x_n)$ where $[\psi] \leq [\phi]$ if $\psi \vdash \phi$ and two formulas are equivalent if they are equiprovable.

Generalized projections

Definition

A class of morphisms Λ of a category $\mathcal{C}$ is said to be a **left class of morphisms** if:

1. given an arrow fh of $\mathcal{C}$, if $f \in \Lambda$ and $h \in \Lambda$, then we have $fh \in \Lambda$;
2. pullbacks of arrows in Λ exist for every arrow of $\mathcal{C}$ and for every $f \in \Lambda$ and g of $\mathcal{C}$, for every pullback square



   we have that $g^*f \in \Lambda$;
3. every isomorphism is in Λ.

Λ-existential doctrines

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** be a conjunctive doctrine and let Λ be a left class of morphisms of $\mathcal{C}$. P is called a Λ-**existential** if, for any arrow $f : A \longrightarrow B$ of Λ, the functor $P_f : P(B) \longrightarrow P(A)$ has a left adjoint $\exists_f$, and these satisfy:

(BC) **Beck-Chevalley condition**;

(FR) **Frobenius reciprocity**.

Notation: When Λ is the class of product projections, we will speak about **pure-existential doctrines**, while when Λ is the class of all the morphisms of the base we will speak about **full-existential doctrines**.

Examples

Example

- $\mathrm{Sub}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** is Mono-existential

$$\exists_f(g) := f \circ g, \text{ with } f \colon X \hookrightarrow Y \text{ and } g \colon Z \hookrightarrow X$$

- $\Psi_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** is full-existential

$$\exists_f(g) := f \circ g, \text{ with } f \colon X \longrightarrow Y \text{ and } g \colon Z \longrightarrow X$$

- $\mathcal{A}^{(-)} \colon \mathrm{Set}^{\mathrm{op}} \longrightarrow$ **InfSL** is full-existential

$$\exists_f(g)(y) := \bigvee_{x \in f^{-1}(y)} g(x), \text{ with } f \colon X \longrightarrow Y \text{ and } g \colon X \longrightarrow \mathcal{A}$$

- $\mathrm{LT}_{=,\exists} \colon \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=,\exists}} \longrightarrow$ **InfSL** is pure-existential

$$\exists_{\mathrm{pr}}(\alpha(x,y)) := \exists x.\alpha(x,y)$$

Existential-free elements

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an Λ-existential doctrine. An object β of the fibre $P(B)$ is said to be a Λ-**existential splitting** if for every morphism $g\colon C \longrightarrow B$ in Λ and for every element γ of the fibre $P(C)$, whenever

$$\beta = \exists_g(\gamma)$$

holds then there exists an arrow $h\colon B \longrightarrow C$ such that $gh = \mathrm{id}$ and

$$\beta = P_h(\gamma).$$

Moreover, an object α of the fibre $P(A)$ is said to be Λ-**existential-free** if, for every morphism $f\colon B \longrightarrow A$, $P_f(\alpha)$ is a Λ-existential splitting.

Examples of existential-free elemets

Example

- The Mono-existential-free elements of the doctrine $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ are exactly the identities arrows;
- the full-existential-free elements of the doctrine $\Psi_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ are exactly the identities arrows;
- the full-existential-free elements of the doctrine $\mathcal{A}^{(-)}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ are exactly the functions $f\colon X \longrightarrow \mathcal{A}$ such that every element $f(x)$ is supercompact, i.e. if $f(x) \leq \bigvee_{i\in I} b_i$ then $f(x) \leq b_{\bar{i}}$ for some $\bar{i} \in I$;
- the pure-existential-free elements of the doctrine $\mathrm{LT}_{=,\exists}\colon \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=,\exists}} \longrightarrow \mathbf{InfSL}$ are exactly the formulae which are free from the existential quantifier.

Enough existential free elements

Definition

We say that a Λ-existential doctrine P has **enough-Λ-existential-free objects** if for every object A of $\mathcal{C}$, any element $\alpha \in P(A)$ is Λ**-covered** by some element $\beta \in P(B)$ for some object B of $\mathcal{C}$, namely β is a Λ-existential-free element and

$$\alpha = \exists_g(\beta)$$

with $g\colon B \longrightarrow A$ arrow of Λ.

Example

- $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** has enough-Mono-existential-free objects;
- $\Psi_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** has enough-full-existential-free objects;
- $\mathrm{LT}_{=,\exists}\colon \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=,\exists}} \longrightarrow$ **InfSL** has enough-pure-existential-free objects;

Generalized existential completion

Given a conjunctive doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and a left class Λ of morphisms of $\mathcal{C}$, we can construct a new Λ-existential doctrine

$$\mathrm{Ex}^{\Lambda}(P)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}.$$

This construction is called **generalized existential completion**. As particular cases, we have the following constructions:

- **full existential completion**: if $\mathcal{C}$ has finite limits, we can construct a full existential doctrine $\mathrm{fEx}(P)$;
- **pure existential completion**: if $\mathcal{C}$ has finite products, we can construct an existential doctrine $\mathrm{pEx}(P)$.

D. Trotta (2020), *The Existential Completion*, Theory and Applications of Categories, 2020, Vol. 35, No. 43.
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Characterization of generalized existential completion

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a* $\wedge$*-existential doctrine. Then the following are equivalent:*

- $P \cong \mathrm{Ex}^{\wedge}(P')$ *for some conjunctive doctrine* $P'\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$;
- P *satisfies the following points:*
  1. P *has enough-*$\wedge$*-existential-free objects;*
  2. P *satisfies* $\wedge$*-RC, i.e. each top element* $\top_A$ *of a fibre* $P(A)$ *is a* $\wedge$*-existential-free object;*
  3. *for every* $\wedge$*-existential-free object* α *and* β *of* $P(A)$, $\alpha \wedge \beta$ *is a* $\wedge$*-existential-free object.*

M.E. Maietti D. Trotta (2023), *A characterization of generalized existential completions*, Annals of Pure and Applied Logic, Vol. 174, No. 4.

Examples of generalized existential completions

Example

- The doctrine $\mathrm{Sub}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is isomorphic to the generalized existential completion $\mathrm{Ex}^{\mathrm{Mono}}(\mathrm{t})$ of the trivial doctrine $\mathrm{t} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, i.e. $\mathrm{t}(X) := \{\bullet\}$ for every object X;
- $\Psi_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is isomorphic to the full existential completion $\mathrm{fEx}(\mathrm{t})$ of the trivial doctrine $\mathrm{t} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$;
- $\mathrm{LT}_{=,\exists} : \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=,\exists}} \longrightarrow \mathbf{InfSL}$ is isomorphic to the pure existential completion $\mathrm{pEx}(\mathrm{LT}_{=})$ of the syntactic doctrine $\mathrm{LT}_{=} : \mathcal{C}^{\mathrm{op}}_{\mathcal{L}_{=}} \longrightarrow \mathbf{InfSL}$ associated with the Horn fragment of first order Intuitionistic logic.
- the localic doctrine $\mathcal{A}^{(-)} : \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is a full existential completion if and only if the locale $\mathcal{A}$ is supercoherent.

More examples of generalized existential completions

Definition

A pure existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is said to be **equipped with** Hilbert's ϵ-**operators** if for every element α in $P(A \times B)$ there exists an arrow $\epsilon_\alpha\colon A \longrightarrow B$ such that

$$\exists_{\mathrm{pr}_1}(\alpha) = P_{\langle \mathrm{id}_A, \epsilon_\alpha \rangle}(\alpha)$$

holds in $P(A)$, where $\mathrm{pr}_1\colon A \times B \longrightarrow A$ is the first projection.

Theorem

A pure existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is equipped with Hilbert's* ϵ*-operators if and only if* $P \cong \mathrm{pEx}(P)$.

More examples of generalized existential completions

Definition

Let $\mathbb{A}$ be a PCA. We define the **realizability doctrine**

$$\mathcal{P}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

The partial ordered set $(\mathcal{P}(X), \leq)$ is defined as follows: let $\mathrm{P}(\mathbb{A})^X$ denote the set of functions from X to the powerset $\mathrm{P}(\mathbb{A})$ of $\mathbb{A}$. Let $\leq$ be the preorder on this set defined as: $\alpha \leq \beta$ if there exists an element $c \in \mathbb{A}$ such that for all $x \in X$ and all $a \in \alpha(x)$ we have that $c \cdot a$ is defined and $c \cdot a \in \beta(x)$. Then $\mathcal{P}(X)$ is defined as the quotient of $\mathrm{P}(\mathbb{A})^X$ by the equivalence relation $\sim$ generated by $\leq$.

Theorem

The realizability doctrine $\mathcal{P}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is isomorphic to the full existential completion of the doctrine* $\mathcal{P}^{\mathrm{sing}}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *of singletons.*

Recap: Dialectica interpretation

Gödel's Dialectica Interpretation: an interpretation of intuitionistic arithmetic HA in a quantifier-free theory of functionals of finite type, called system T.

Idea: translate every formula A of HA to $A^D = \exists x \forall y A_D$, where A_D is quantifier-free.

Application: if HA proves A, then system T proves $A_D(t, y)$, where y is a string of variables for functionals of finite type, and t a suitable sequence of terms (not containing y).

Goal: to be as constructive as possible, while being able to interpret all of classical arithmetic.

Gödel (1958), *Über eine bisher noch nicht benützte erweiterung des finiten standpunktes*, Dialectica, 12(3-4):280-287.

Recap: Dialectica interpretation

The most complicated clause of the translation is the definition of the translation of the **implication connective** $(\psi \to \phi)^D$:

$$(\psi \to \phi)^D = \exists f_0, f_1. \forall u, y. (\psi_D(u, f_1(u, y)) \to \phi_D(f_0(u), y)).$$

This involves three logical principles: a form of the **Principle of Independence of Premise** (IP), a generalisation of **Markov Principle** (MP), and the **axiom of choice** (AC).

Intuition: given a witness u for the hypothesis ψ_D, there exists a function f_0 assigning a witness $f_0(u)$ of ϕ_D to every witness u of ψ_D. Moreover, this assignment has to be such that from a counterexample y of the conclusion ϕ_D we should be able to find a counterexample $f_1(u, y)$ to the hypothesis ψ_D.

Gödel, Feferman, et al (1986), *Kurt Gödel: Collected Works: Volume II:*, Oxford University Press.

First-order hyperdoctrines

Definition

A **first-order hyperdoctrine** is a contravariant functor:

$$P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Hey}$$

from a category with finite products $\mathcal{C}$ to the category of Heyting algebras **Hey** satisfying:

1. for every projection $\mathrm{pr}_A\colon A \times B \longrightarrow A$, the morphism $P_{\mathrm{pr}_A}\colon P(A) \longrightarrow P(A \times B)$ has a left adjoint $\exists_{\mathrm{pr}_A}$ and a right adjoint $\forall_{\mathrm{pr}_A}$, satisfying (BC).
2. For every object A of $\mathcal{C}$ there exists a predicate δ_A of $P(A \times A)$ satisfying for every α of $P(A \times A)$ that:

   $$\top \leq P_{\Delta_A}(\alpha) \text{ if and only if } \delta_A \leq \alpha$$

   where $\Delta_A\colon A \longrightarrow A \times A$ denotes the diagonal arrow.

Definition

A first-order hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Hey}$ is called a **Gödel hyperdoctrine** if:

1. the category $\mathcal{C}$ is cartesian closed;
2. the doctrine P has enough pure-existential-free predicates;
3. the pure-existential-free objects of P are stable under universal quantification, i.e. if $\alpha \in P(A)$ is existential-free, then $\forall_{\mathrm{pr}}(\alpha)$ is pure-existential-free for every projection pr from A;
4. the sub-doctrine P' of the pure-existential-free predicates of P has enough pure-universal-free predicates.

An element α of a fibre $P(A)$ of a Gödel hyperdoctrine P that is both an existential-free predicate and a universal-free predicate in the sub-doctrine P' of existential-free elements of P is called a **quantifier-free predicate** of P.

D. Trotta, M. Spadetto and V. de Paiva (2023), *Dialectica Principles via Gödel Doctrines*, Theoretical Computer Science, Vol. 947.

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Notation. From now on, we shall employ the logical language provided by the **internal language** of a doctrine and write:

$$a_1 : A_1, \ldots, a_n : A_n \mid \phi(a_1, \ldots, a_n) \vdash \psi(a_1, \ldots, a_n)$$

instead of:

$$\phi \leq \psi$$

in the fibre $P(A_1 \times \cdots \times A_n)$. Similarly, we write:

$$a : A \mid \phi(a) \vdash \exists b : B. \psi(a, b) \text{ and } a : A \mid \phi(a) \vdash \forall b : B. \psi(a, b)$$

in place of:

$$\phi \leq \exists_{\mathrm{pr}_A} \psi \text{ and } \phi \leq \forall_{\mathrm{pr}_A} \psi$$

in the fibre $P(A)$. Also, we write $a : A \mid \phi \dashv\vdash \psi$ to abbreviate $a : A \mid \phi \vdash \psi$ and $a : A \mid \psi \vdash \phi$.

Skolemisation in Gödel hyperdoctrines

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Hey}$ *be a Gödel hyperdoctrine, and let* α *be an element of* $P(I)$*. Then there exists a quantifier-free predicate* α_D *of* $P(I \times U \times X)$ *such that:*

$$i : I \mid \alpha(i) \dashv\vdash \exists u : U.\forall x : X.\alpha_D(i, u, x).$$

Theorem

Every Gödel hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Hey}$ *validates the* ***Skolemisation principle****, that is:*

$$a_1 : A_1 \mid \forall a_2.\exists b.\alpha(a_1, a_2, b) \dashv\vdash \exists f.\forall a_2.\alpha(a_1, a_2, fa_2)$$

where $f : B^{A_2}$ *and* fa_2 *denote the evaluation of* f *on* a_2*, whenever* $\alpha(a_1, a_2, b)$ *is a predicate in the context* $A_1 \times A_2 \times B$*.*

Principle of Independence of Premise in Gödel hyperdoctrines

Theorem

Every Gödel hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **Hey** *satisfies the* ***Rule of Independence of Premise****: whenever* $\beta \in P(A \times B)$ *and* $\alpha \in P(A)$ *is a existential-free predicate, it is the case that:*

$$a : A \mid \top \vdash \alpha(a) \to \exists b.\beta(a,b) \text{ implies that } a : A \mid \top \vdash \exists b.(\alpha(a) \to \beta(a,b)).$$

Moreover, if existential-free predicates are closed with respect to finite conjunctions, then P *satisfies the* ***Principle of Independence of Premise****:*

$$a : A \mid \top \vdash (\alpha(a) \to \exists b.\beta(a,b)) \to \exists b.(\alpha(a) \to \beta(a,b)).$$

D. Trotta, M. Spadetto and V. de Paiva (2023), *Dialectica Principles via Gödel Doctrines*, Theoretical Computer Science, Vol. 947.

Modified Markov Principle in Gödel hyperdoctrines

Theorem

Every Gödel hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **Hey** *satisfies the following* ***Modified Markov Rule****: whenever* $\beta_D \in P(A)$ *is a quantifier-free predicate and* $\alpha \in P(A \times B)$ *is an existential-free predicate, it is the case that:*

$$a : A \mid \top \vdash (\forall b.\alpha(a,b)) \to \beta_D(a) \text{ implies that } a : A \mid \top \vdash \exists b.(\alpha(a,b) \to \beta_D(a)).$$

Moreover, if existential-free predicates are closed with respect to implication, then P *satisfies the following* ***Modified Markov Principle****:*

$$a : A \mid \top \vdash (\forall b.\alpha(a,b) \to \beta_D(a)) \to \exists b.(\alpha(a,b) \to \beta_D(a)).$$

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Markov Principle in Gödel hyperdoctrines

Corollary

*Every Gödel hyperdoctrine $P : \mathcal{C}^{op} \longrightarrow$ **Hey** such that $\bot$ is a quantifier-free predicate satisfies **Markov Rule**: for every quantifier-free element $\alpha_D \in P(A \times B)$ it is the case that:*

$$b : B \mid \top \vdash \neg \forall a. \alpha_D(a, b) \text{ implies that } b : B \mid \top \vdash \exists a. \neg \alpha_D(a, b).$$

*Moreover, if existential-free predicates are closed with respect to implication, then P satisfies **Markov Principle**:*

$$b : B \mid \top \vdash \neg \forall a. \alpha_D(a, b) \rightarrow \exists a. \neg \alpha_D(a, b).$$

D. Trotta, M. Spadetto and V. de Paiva (2023), *Dialectica Principles via Gödel Doctrines*, Theoretical Computer Science, Vol. 947.

Strengthened Dialectica functionals

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **Hey** *be a Gödel hyperdoctrine such that the existential-free predicates are closed with respect to implication and finite conjunctions and falsehood* $\bot$ *is a quantifier-free predicate. Then for every* ψ_D *in* $P(I \times U \times X)$ *and* ϕ_D *in* $P(I \times V \times Y)$ *quantifier-free predicates of* P *we have that the formula:*

$$i : I \mid \exists u.\forall x.\psi_D(i,u,x) \rightarrow \exists v.\forall y.\phi_D(i,v,y)$$

is provably equivalent to:

$$i : I \mid \exists f_0, f_1.\forall u, y.(\psi_D(i,u,f_1(i,u,y)) \rightarrow \phi_D(i, f_0(i,u), y)).$$

D. Trotta, M. Spadetto and V. de Paiva (2023), *Dialectica Principles via Gödel Doctrines*, Theoretical Computer Science, Vol. 947.

Future work

- Characterize exact and regular completions of doctrines (j.w.w. M. Maietti);
- Abstract the sheafification as tripos-to-topos-adjunction (j.w.w. M. Maietti);
- Abstract the notion of "supercampact element" to implicative algebras, and characterize their assemblies (j.w.w S. Maschio);
- Categorify computational notions: Medvedev, Muchnik and Weihrauch reducibilities (j.w.w. M. Valenti and V. de Paiva)