# *AS IF* MATHEMATICS WERE TRUE

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# AIMS

- Claim: An *as-ifist* interpretation of mathematics can be used to provide an account of both the *practice* and the *applicability* of mathematics whilst avoiding the confusion of mathematical and metaphysical considerations.
  - I begin first with Plato to show that much philosophical milk has been spilt owing to our conflating the method of mathematics with the method of philosophy.
  - I further use my reading of Plato to develop what I call *methodological as-ifism*, the view that, in mathematics, we treat our hypotheses *as if* they were first principles and we do this for the purpose of solving mathematical problems.
  - I next extend *as-ifism* to modern mathematics wherein the method of mathematics becomes the axiomatic method, noting that this engenders a shift from as-if hypotheses to as-if axioms, and a structuralist shift from the investigation of objects themselves to the investigation of objects as positions in a structure.
  - I pause to note that the conflation of the method of mathematics with the method of philosophy, witnessed by the Frege-Hilbert debate, has led to the continued confusion of mathematics with metaphysics.
  - Finally, I use my methodological as-ifism to reconsider the structuralist foundations debate, specifically, that between using set-theory or category-theory?

# PLATO

- Plato kept a clear distinction between mathematics and metaphysics and the knife he used to slice the difference between the two was method.
  - The mathematical method reasons *down from an hypothesis* towards a conclusion, with the purpose of solving a *mathematical* problem.
    - The soul is *forced to* use hypotheses in the investigations of [a problem], not traveling up to a first principle, since *it cannot escape* or get above its hypotheses… (511a)

  - The philosophical method reasons *up from an hypothesis* towards an unhypothetical first principle which tethers the hypothesis by fixing it to a form, and only then can they reason down towards a conclusion, with the purpose of solving a *philosophical* problem.
    - Also understand, then, that by the other subsection of the intelligible I mean what reason itself grasps by the power of *dialectical discussion*, treating its hypotheses, not as first principles, but as *genuine hypotheses*, that is, stepping stones and links in a chain, in order to arrive at what is *unhypothetical* and the *first principle* of everything. Having grasped this principle, it reverses itself and, keeping hold of what follows from it, comes down to a conclusion…moving on through forms to forms, and ending in forms.(511b-c)

# PLATO

- The mathematician's hypotheses are taken *as if* they were first principles, but they are not, the mathematician's objects are taken *as if* they were stable objects of knowledge, but they are not.
  - students of geometry, calculation, and the like *hypothesize* the odd and the even, the various figures, the three kinds of angles, and other things akin to these in each of their investigations, *regarding them as known*. These they treat these as if they were first principles and *do not think it necessary to give any account of them*, either to themselves or to others, *as if* they were evident to everyone. And, consistently going from these first principles through the remaining steps, they conclude in full agreement at *the point they set out to reach in their investigation*. (510c-d)
- The purpose of the mathematicians' method, which *begins* with *taking* hypotheses *as if* they were true first principles, is to *solve a mathematical problem*, it is not to give a philosophical account of their truth by fixing them to a domain of stable objects, i.e., to independently existing forms or physical objects.
  - Just as in geometry, then, *it is by making use of problems*, that we will pursue astronomy too. We will leave the things in the heavens alone if we are really going to participate in astronomy and make the naturally wise element in the soul useful instead of useless. (530b)

# PLATO

- Thus, we come to Plato's *methodological as-ifism*: we treat our hypotheses *as if* they are true and our objects *as if* they exist for the purpose of solving a mathematical problem.

- In solving the *Meno's* mathematical problem, I treat the length of line that doubles the area of a two-unit square *as if* were a stable object, but it is not. Moreover, it is only because of the precision of the definitions of square and of diagonal that I can consistently reason down to the conclusion that the length will be the measure of the diagonal of the four-unit square, *but* I cannot *know* the length of this line as itself a stable object since it is $2\sqrt{2}$

- In solving the *Republic's* meta-mathematical problem, of "what all mathematical objects have in common", I treat mathematical objects themselves as if they were precisely defined by a geometric theory of proportion, so I "can draw conclusions about their kindship", for example, so that I can precisely define numbers themselves as lengths or measures of geometric ratios, *but* I cannot *know* numbers as stable objects.

# PLATO

- The *hypothetical* method of mathematics is distinct *from* the *metaphysical* method of philosophy, and, as such, so is its ontology and epistemology.
  - Mathematical objects are not objects of knowledge, they not as real as philosophical objects, but, as *objects of thought*, they are still "concerned with being" (534a).
  - The mathematical method yields a kind of understanding but not knowledge, that is, it yields beliefs that are "reliable guides to solving problems" (532b) because they are born out of *precise definitions* and a *stable method*.
  - Only the metaphysical method of philosophy yields true understanding or knowledge, that is, yields true beliefs that are themselves tethered to first principles, that are fixed by a domain of *stable objects* or forms.

- Against metaphysical realism (platonism): mathematics does not need a domain of stable objects, a metaphysics of forms or a mathematical foundation, it only needs *precise definitions* and a *stable method*.
  - At the *object-level* mathematical objects are taken at face value as precisely defined by pure mathematical theories.
  - At the *meta-level* mathematical objects themselves as "kinds of objects" are precisely defined by the geometric theory of proportion.

# PLATO'S METHODOLOGICAL AS-IFISM

- In mathematics, both at the object-level and at the meta-level, we treat our hypotheses *as if* they were true first principles, and, *consequently*, our objects *as if* they exist, and we do this with the purpose of solving mathematical problems.

- Mathematics as a "science" is founded on the precision of its definitions and the stability of the hypothetical method.

- The *confusion*: mathematics is *not* a science founded on the dialectic method, wherein the truth of its first principles is fixed by the stability of metaphysical or foundational objects.

# CORRECTING THE CONFUSION

- The continued confusion: structuralist philosophers of mathematics, have continued to conflate the hypothetical method of mathematics with the metaphysical method of philosophy and so have made structures into objects.
  - They continue to take their background meta-mathematical theories of mathematical structures (structure theory (Shapiro), system theory (Hellman), Set theory, Category Theory, etc.) as metaphysical or foundational, i.e., as providing a domain of stable objects that fix the truth of their axioms.

- The correction: as with Plato, we should take our background meta-mathematical theory of structure a) as mathematical (not metaphysical) and b) as providing precise definitions and stable methods (not stable objects), so we can act *as if* our axioms were first principles, and this for the purpose of solving the meta-mathematical problem of how to talk about mathematical structures themselves.
  - At the *object-level*, mathematical objects are taken at face value as precisely defined by an axiomatic mathematical theory; e.g., a group as an object, is a position in a group structure, i.e., is whatever satisfies the group axioms.
  - At the *meta-level*, mathematical structures themselves as "kinds of structures" are precisely defined by category theory.

# MODERN AS-IFISM

- To provide the details of this correction, I now extend methodological as-ifism to modern mathematics wherein the method of mathematics becomes the axiomatic method.

- This engenders two shifts [Burgess, 2015]
  - A shift from starting with as-if hypotheses to starting with as-if axioms, i.e., to starting with taking axioms as if they were true first principles,
  - A structuralist shift from the investigation of objects themselves to the investigation of objects as positions in a structure.

- Mathematics as a science is founded on the stability of the axiomatic method and the precision of its definitions, now as *implicitly expressed by the axioms themselves*; again, it is *not* founded on taking axioms as true first principles and so on the stability objects themselves, either mathematical or metaphysical objects.

- We see this object-level confusion play out with the Frege-Hilbert debate.

# + THE FREGE-HILBERT DEBATE

- Frege, for example, confuses the method of mathematics with the method of philosophy (with the method of concept construction), that is, he takes axioms as true first principles and so presumes that we first need a stable domain of objects themselves to fix their truth.
  - For the Fregean axioms-as-first-principles account, the primitive terms (concepts) employed by the axioms must be *explicitly defined* over a *fixed* domain of objects *before* the statement of the axioms.

- Hilbert, by contrast, takes axioms *as if* they were true first principles that is, they *implicitly define* the primitive terms (concepts), so whatever satisfies the axioms is taken as an object that fixes the truth of the axioms.
  - Hilbert took axioms as *implicit* definitions over a *variable* domain, so that the axioms systems themselves are but a "*schema*" for defining those concepts that organize what we say about the objects as *variously interpreted*.

# METAMATHEMATICAL AS-IFISM

- For Frege the stability of mathematical definitions and the precision of its method was to be justified by assuming the truth of the axioms, truth as fixed *logically*, in the case of arithmetic, or truth as fixed *philosophically* by Kantian intuition, in the case of geometry.
  - Frege's meta-mathematical account of the method of mathematics was: *if* the axioms are true, *then* this theorem can be justified, So his focus was on establishing the truth of his axioms.
- For Hilbert, however, the stability of mathematical definitions and the precision of its method was justified by assuming the *consistency* of the axioms. Hence Hilbert's famous quote:
  - *if* the arbitrary postulated axioms do not contradict each other with their collective consequences, *then* they are true and the things defined by means of the axioms exist. That, for me, is the criterion of truth and existence.
  - Hilbert's meta-mathematical account of the method of mathematics was: *if* the axioms are consistent, *then* this theorem can be justified, So his focus was on establishing the consistency of his axioms.

# IF-THENISM VERSUS AS-IFISM

- Fearing a collapse to formalism, Frege came to reject *logical if-thenism* as a meta-mathematical account of the stability of method of mathematics.

- Gödel's results, likewise, collapsed Hilbert's *logical if-thenist* programme.

- What I will now consider is whether, given the *structuralist shift* from objects themselves to objects as positions in a structure, these *logical if-thenist* views can reconsidered in terms of the *methodological as-ifist* view that Plato seemed to be offering up.

# LOGICAL IF-THENISM

- According to Resnik, Frege developed two forms of if-thenism.
- On the first *deductive if-thenist* option, mathematics is in the business of establishing results in pure logic.
  - This first option can either be expressed as "$\mathbf{A} \rightarrow \mathbf{T}$" is logically valid (logically provable) or as the claim that $\mathbf{A} \vdash \mathbf{T}$ ($\mathbf{T}$ is logically derivable from $\mathbf{A}$).
- On the second *structural if-thenist option*, Frege
  - views a mathematical theory as studying the properties of all structures *satisfying* certain defining conditions, but he never makes use of the assumption that such structures exist [Resnik, 1980, 117].
  - This option is expressed as "$\mathbf{A} \vDash \mathbf{T}$" ($\mathbf{T}$ is logically entailed by $\mathbf{A}$).

# STRUCTURAL IF-THENISM

- Resnik next notes that the second option offers a straightforward path to a *structuralist account* of
  - a) mathematical objects as positions in a structure, i.e., a natural number is a position in any or all structures *satisfying* the Peano-Dedekind axioms;
  - b) mathematical applicability
    - when one finds a physical structure *satisfying* the axioms of a mathematical theory, the application of that theory is immediate [Resnik, 1980, 118]
- Resnik further mentions a final historical virtue, viz., that
    - such a structural approach is in-line with the development of abstract structures, like group theory and topology [Resnik, 1980, 118] … and category theory!

# PROBLEMS WITH STRUCTURAL AS-IFISM

- The Structure Problem (Hellman's "Home Address" Problem)
  - We need set theory or some other "foundational" theory as a semantic or ontological background meta-mathematical *theory of structures themselves.*
    - *Hellman's modal structuralist account of possible concrete structures.*
    - *Shapiro's platonist account of actual abstract structures.*

- The Consistency Problem
  - Vacuity problem (If-thens are made trivially true by false antecedents).
  - All inconsistent theories will define the same structure.
  - One must assume that the majority of mathematical theories are consistent.

# PROBLEMS WITH STRUCTURAL AS-IFISM

- Faced with these problems, Resnik presents us with two meta-mathematical alternatives:

  - 1. We can take the *Fregean* route of *turning to philosophy* and base the assumption of consistency on
    - a belief in the mathematical *reality* and so the *truth* of some theory (of structures) which will vouch safe the consistency of mathematical theories [Resnik, 1980, 119].

  - 2. We take the *Carnapian* route of *turning to logic* and
    - begin with (a mathematical theory as) a linguistic framework which is referential . . . and thus agrees with the *prima facie* referential character of mathematical language as used by practicing mathematicians'… and argue that since consistency is a mathematical question, it, too must be treated logically…[so] the assertion that a given axiom set is consistent must itself be construed as conditional upon a background theory *with respect to whose truth the deductivist can remain agnostic* [Resnik, 1980, 119].

# METHODOLOGICAL STRUCTURAL AS-IFISM

- Resnik, Shapiro, Hellman, and most set-theoretic foundationalists, take the Fregean metaphysical route, I take the Carnapian but, instead of turning to logical rules in a deductive if-thenist context, I turn to mathematical methods in a structural as-ifist context.

- That is, I use category theory as my meta-mathematical background theory for talk of structures themselves as "kinds of structures", e.g., **EM** axioms for talk of object-level structures, like groups, topological spaces, the **ETCS** Axioms for talk of set-structures, and **CCAF** axioms for talk of categories, and I leave the choice of the logical rules/methods open
  - This meta-mathematical theory is not taken as a metaphysical theory that fixes the truth of the axioms by making structures into objects; it is not "about" objects and arrows, set and functions, or categories and functors, so there is no "structure" (or "home address") problem.
  - It is taken as a Carnapian linguistic framework, and further its axioms are taken as *schematic* in the Hilbertian sense that the 'object' and 'arrow', 'set' and 'function', or 'category' and 'functor', are themselves taken as *ranging over a variable domain*.
  - Finally, we act *as if* it were a consistent mathematical theory for the methodological purpose of providing a language for mathematical structuralism, so there is no "consistency" problem… maybe!

# METHODOLOGICAL STRUCTURAL AS-IFISM

- Let's now pause to compare *metaphysically* interpreted structuralism with *methodologically interpreted* structural to see if we can finally forestall Shapiro's latest version of the "consistency" problem, namely, that on such an Hilbertian or "algebraic" approach
  - The possible infinite regress of relative consistency proofs will only be stopped by a *true* meta-mathematical theory; otherwise, we will need to turn to *logic*, and make use of a completeness theorem, or to *philosophy*, and concede that consistency is not a logical notion. [Shapiro, 2005, 70]

- Notice that on my *as-if methodological* account, to solve the meta-mathematical problem of the possible regress of relative consistency proofs, we need only act *as if* some theory is true *for this meta-mathematical purpose*.

- To think that it needs to be taken as unconditionally true (in either a metaphysical or foundational sense) is not something I only remain agnostic about, it is something I outright reject, because it conflates the method of mathematics with the method of philosophy!

# METHODOLOGICAL STRUCTURAL AS-IFISM

- The proposed *methodological structural as-ifist* position holds that some of our methodological commitments to taking our axioms *as if* they were first principles,
  - some will be made in light of *mathematical practice*, with the goal of solving mathematical problems,
  - some will be made in light of *mathematical applicability*, with the goal of solving physical problems, and,
  - some will be made in light of *logical/philosophical considerations*, with the goal of solving meta-mathematical problems.
- None of these commitments, however, will be made with the goal of solving metaphysical problems, i.e., problems about what "fixes" truth or "fixes" consistency.

# + AS-IFISM *AND* PLURALISM

- With respect to these logical/philosophical considerations:
  - Does this mean that we will we have to call in a logic/a model-theory to explain what we mean by satisfaction? Yes.
  - Do we need meta-theory of ‘models’ themselves? Yes.
  - Does this mean that we will have to take models themselves as naturalistically constructed (Maddy) or as modally interpreted (Putnam)? No.
  - Does it have to be a first order logic? No.
  - Does it have to be a classical logic? No.

# AS-IFISM *AND* PLURALISM

- Do we need a meta-theory of ‘structures’ themselves? Yes.
- Does this mean that we will have to take structures themselves as actually existing (Shapiro) or possibility existing (Hellman)? No.
- Can it be set theory? Yes.
- Does it have to be set theory? No.
- Can it be category theory? Yes.
- Does it have to be category theory? No.

# CATEGORY THEORY AS A LANGUAGE

- So why do I advocate taking CT as *the* linguistic framework for mathematical structuralism?
  - 1. We can give an as-if methodological reading of **ST** but it will not be schematic, we will still have a fixed domain (of sets for **ZFC**, classes for **GB**, urelements for **ZFA**) as either *ontologically or semantically prior*; for **CT**, the axioms need only be taken as *definitionally prior*.
  - 2. **CT** takes the objects of mathematics at face value; because 'objects' and 'arrows' are taken as schematic, as with Mac Lane, as 'undefined terms or predicates' [Mac Lane, 1968, 287], the structures it talks about, e.g., sets, groups, topological spaces, deductive systems, can be taken at face value and so do not require a "reduction to" set structure, again, in the sense of either an ontological or semantic reduction.
  - 3. **CT** does better at capturing the shared structure of the various kinds of structures in terms of functors, identity maps, categorical equivalence, etc., that is, we are not restricted merely to isomorphism and the many problems this notion brings.

# CATEGORY THEORY *AS IF* IT WERE A FOUNDATION

- Conclusion: when answering the question
  - Wherein lie the meta-level conditions for speaking about structures themselves?

- We are methodologically committed to taking our CT axioms *as if* they were consistent, this allows us to methodologically act *as if* category theory were a foundation for mathematical structuralism.

- Yet, as with Plato, we all the while realize that, metaphysically speaking, it is not, nor should it be!