U.S. AIR FORCE

ACT3

USSF

AFRL

Applying Categorical Thinking to Practical Domains

Jared Culbertson

Air Force Research Laboratory - Autonomous Capabilities Team (ACT3)

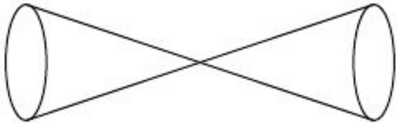
(Joint work with Kirk Sturtz, Dan Guralnik, Jakob Hansen, Dan Koditschek, Paul Gustafson, Peter Stiller and influenced by conversations with many other researchers)

Background: Algebraic geometry & Representation theory

G reductive complex algebraic group with Lie algebra $\mathfrak{g}$

$$\mathcal{N} = \{x \in \mathfrak{g} \mid x \text{ is nilpotent in every representation of } \mathfrak{g}\}$$

Example: $\mathfrak{g} = \mathfrak{sl}_2$; $\mathcal{N} = \{2 \times 2 \text{ matrices with zero trace and determinant}\}$



Proved theorems like...

Theorem (C. 2010)

Given functors $\mathcal{D}' \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{D}''$ and t-structures on $\mathcal{D}'$ and $\mathcal{D}''$ such that

- (G1) j^* is essentially surjective
- (G2) there is a t-structure on $\widetilde{i_*\mathcal{D}'}$ induced by the one on $\mathcal{D}'$
- (G3) the t-structures on $\mathcal{D}'$ and $\mathcal{D}''$ are compatible
- (G4) $j^* f = 0 \implies f$ factors through $\widetilde{i_*\mathcal{D}'}$
- (G5) $\widetilde{i_*\mathcal{D}'} = \{X \mid j^* X = 0\}$

then there is a unique t-structure on $\mathcal{D}$ extending those on $\mathcal{D}'$ and $\mathcal{D}''$

Theorem (C. 2010)

For each orbit $C \subset \mathcal{N}$, there is a fully faithful functor

$$\mathcal{IC}(C, -) : \mathcal{M}_{\text{Poi}G}(C) \to \mathcal{M}_{\text{Poi}G}(\mathcal{N})$$

and every simple perverse Poisson sheaf is of the form $\mathcal{IC}(C, \mathcal{F})$

Theorem (C. 2010)

For a Poisson sheaf $\mathcal{F} = \mathcal{O}_C \otimes \mathcal{L}$ on an orbit C, we have

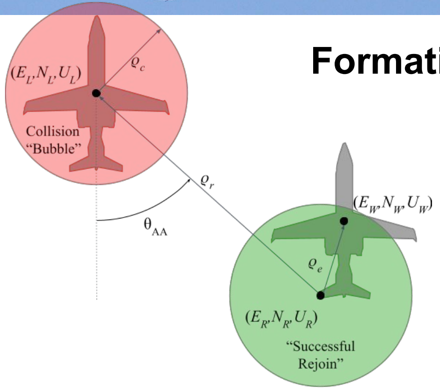
$$i_{C'}^{!\text{Poi}} H^i(\mathcal{IC}(C, \mathcal{F})) = \mathcal{O}_{C'} \otimes \mathcal{L}''$$

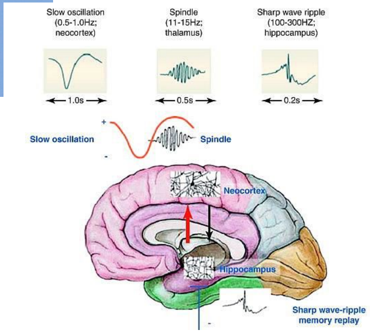
[C., Perverse Poisson Sheaves on the Nilpotent Cone, LSU, 2010]

AFRL - Autonomous Capabilities Team

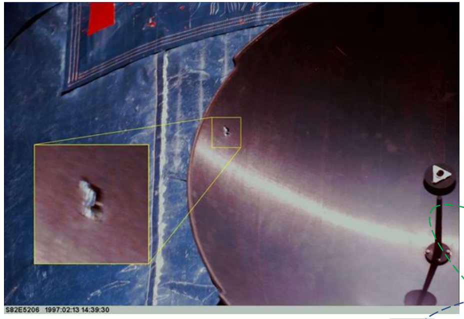
X-62A Modified F-16 (VISTA)

Formation Flying

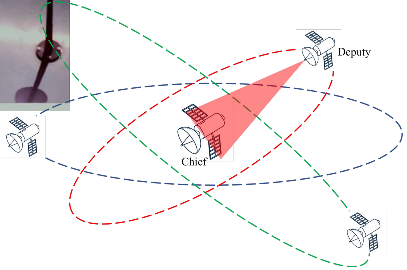




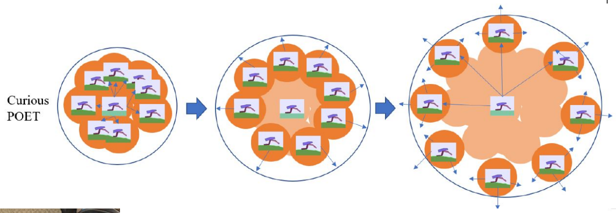
Neuro-mechanisms and architectures

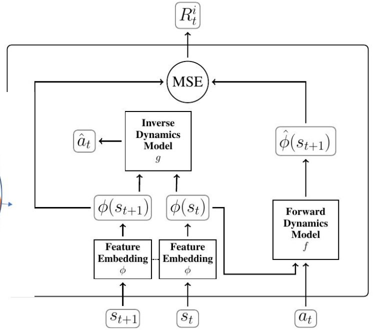


Images: NASA



Multi-agent Spacecraft Inspection





Intrinsic Curiosity & Open-ended Learning

Categorical Probabilities (1960s+ : Lawvere, Huber, Giry, Doberkat, …)

The Probability Monad $\mathcal{P}\colon \mathcal{M}eas \to \mathcal{M}eas$

$$
\begin{array}{lccc}
:_{ob} & X & \mapsto & \mathcal{P}X = \{\text{all probability measures on } X\} \\
:_{ar} & X \xrightarrow{f} Y & \mapsto & \underbrace{\mathcal{P}X \xrightarrow{\mathcal{P}f} \mathcal{P}Y}_{P \mapsto P f^{-1}}
\end{array}
$$

Kleisli Category of $\mathcal{P}$

$\mathbf{Meas}_{\mathcal{G}}$

- Objects: measurable spaces
- Morphisms: stochastic kernels $X \times \Sigma_Y \to [0,1]$
- Composition: given $X \xrightarrow{f} Y \xrightarrow{g} Z$

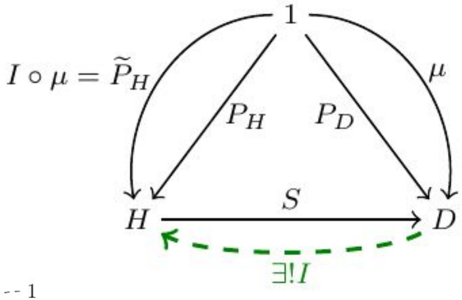
$$(g \circ f)(x, C) = \int_Y g_C \, df_x$$

(with K. Sturtz)

Many results of the flavor: “ * can be expressed as…”

Probability Theory	$\mathcal{M}eas_{\mathcal{G}}$
Probability measure P	Morphism $1 \xrightarrow{P} X$
Measurable function $X \xrightarrow{f} Y$	$x \mapsto \delta_{f(x)}$ = Dirac measure
U-valued random variable	Composition $1 \xrightarrow{P} \Omega \xrightarrow{\delta_X} U$
Regular conditional probability	$\mathcal{M}eas_{\mathcal{G}}$ morphism
Conditional expectation	Diagram in $\mathcal{M}eas_{\mathcal{G}}$
Stochastic process	Functor $\mathcal{T} \to \mathcal{M}eas_{\mathcal{G}}$

Theorem: (C, Sturtz, 2013)
Bayesian probability can be realized in $\mathbf{Meas}_{\mathcal{G}}$



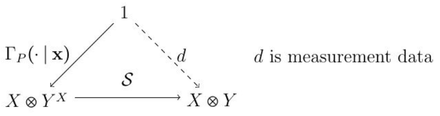


Figure 17: The generic nonparametric Bayesian model for stochastic processes.

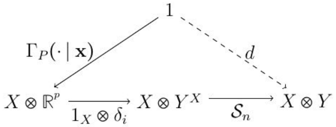
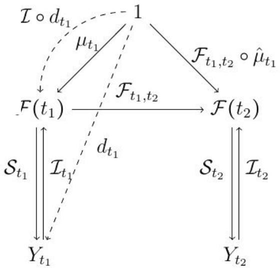


Figure 25: The generic parametric Bayesian model.



[C., Sturtz, A Categorical Foundation for Bayesian Probability, ACS, 2014]

[C., Sturtz, Bayesian Machine Learning via Category Theory, arXiv, 2013]

Figure 40: The hidden Markov model viewed in $\mathcal{P}$.

Original work

- Extremely useful for me in translating to a language I understood
- Not particularly immediately impactful on algorithmic development

Unsatisfying conclusions, but recently…

A free energy principle for generic quantum systems

Chris Fields[a*], Karl Friston[b], James F. Glazebrook[c,d] and Michael Levin[e]

[a] *23 Rue des Lavandières, 11160 Caunes Minervois, FRANCE*
[b] *Wellcome Centre for Human Neuroimaging, University College London, London, WC1N 3AR, UK*
[c] *Department of Mathematics and Computer Science, Eastern Illinois University, Charleston, IL 61920 USA*
[d] *Adjunct Faculty, Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA*
[e] *Allen Discovery Center at Tufts University, Medford, MA 02155 USA*

January 3, 2022

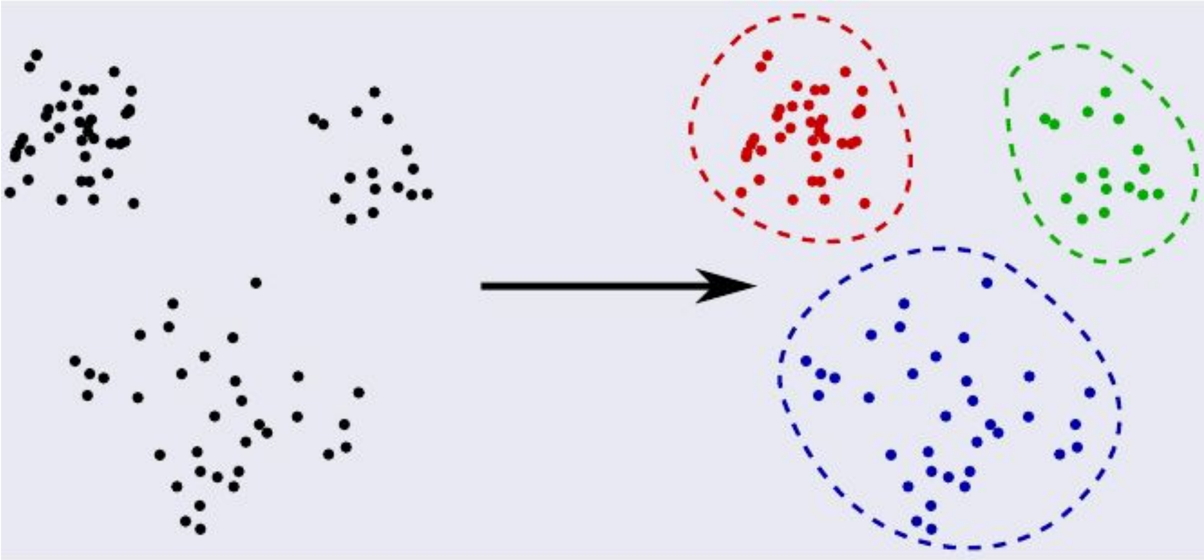
Category Theory in Machine Learning

Dan Shiebler, University of Oxford, daniel.shiebler@kellog.ox.ac.uk
Bruno Gavranović, University of Strathclyde, bruno@brunogavranovic.com
Paul Wilson, University of Southampton, paul@statusfailed.com

Conditional Distributions for Quantum Systems

Arthur J. Parzygnat
Institut des Hautes Études Scientifiques
Bures-sur-Yvette, France
parzygnat@ihes.fr

Data Clustering

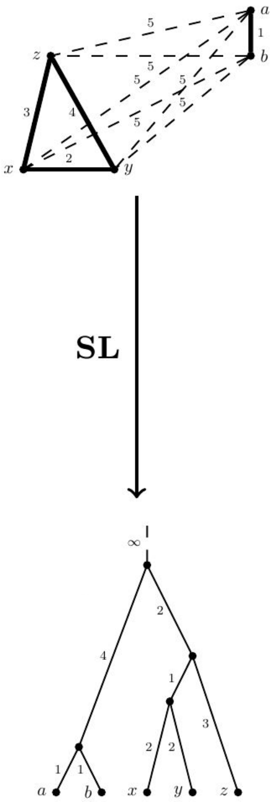


Kleinberg (2002): There are no *consistent* methods of clustering with partitions

- Rich (all partitions realizable)
- Scale-invariant
- Intra/inter-cluster distance transformation invariant

Hierarchical Clustering

Category of Metric Spaces



Category of Ultrametric Spaces

Morphisms: non-expansive maps

$$d(f(x), f(y)) \leq d(x, y)$$

Carlsson–Memoli (2008): There is a unique *functorial* method (single-linkage clustering) for assigning a hierarchical clustering to a metric space.

Q: Are there other useful categories of metric spaces with meaningful projections?

(with D. Guralnik, J. Hansen, P. Stiller)

Theorem (C., Guralnik, Stiller, 2016):

There is an equivalence of categories $\mathbf{Weight} \simeq \mathbf{Sieve}$.

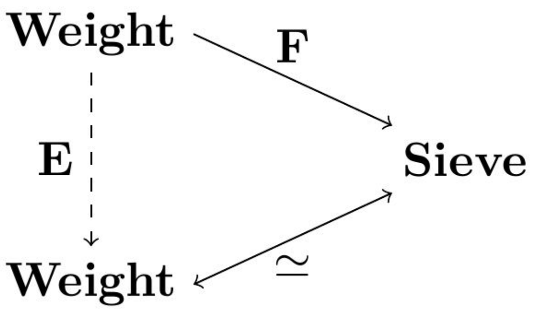
Fibers are directed complete posets under ordering induced from identity maps

{ "nice" subcategories of **weight spaces** }

↕

{ "stationary" structuring maps to **sieves** }

Composition with functor to Weight satisfies an idempotency condition



Corollary: *There is no functorial clustering projection to cut metrics*

[C., Guralnik, Hansen, Stiller, Consistency Constraints for Overlapping Data Clustering, arXiv, 2016]
[C., Guralnik, Stiller, Functorial Hierarchical Clustering with Overlaps, DAM, 2018]

Reality: almost no one uses this theory in practice

Did not address:

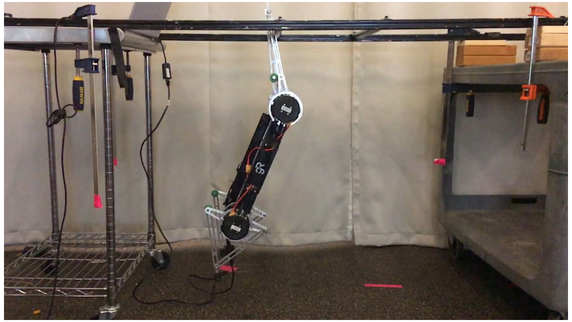
- Cluster uncertainties
- Outlier detection/removal
- **Algorithm efficiency**
- Choice of underlying metrics
- **Software implementations**
- Convincing examples of value
- Characterizations of useful clustering domains

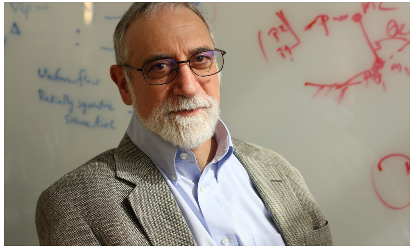
Mismatch with practitioners' needs

- Opaque categorical language with no clear interface for users
- **Consistency in this sense is rarely considered in applications**
- Existing tools are often sufficient for exploratory analysis
- Simplifications were guided by mathematical considerations

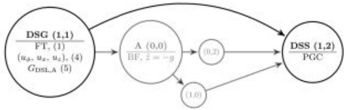
Lesson: "Nice theory → Applications" rarely brings near-term value

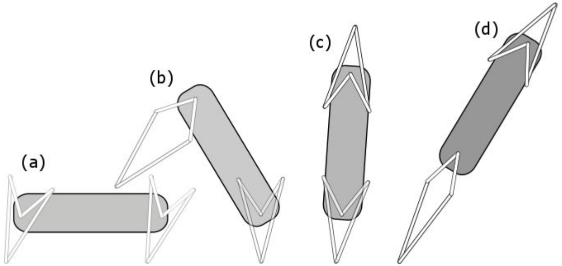
Compositional problems in robotics

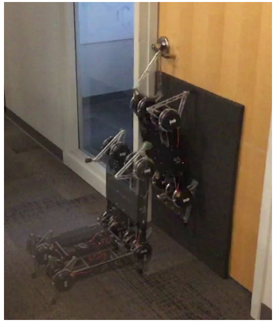


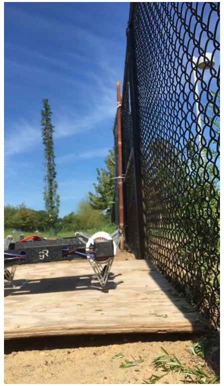


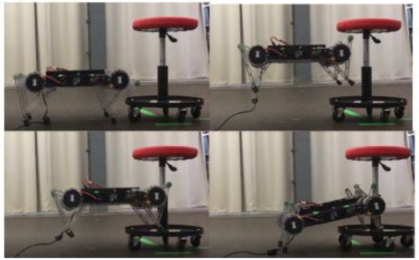
Dan Koditschek





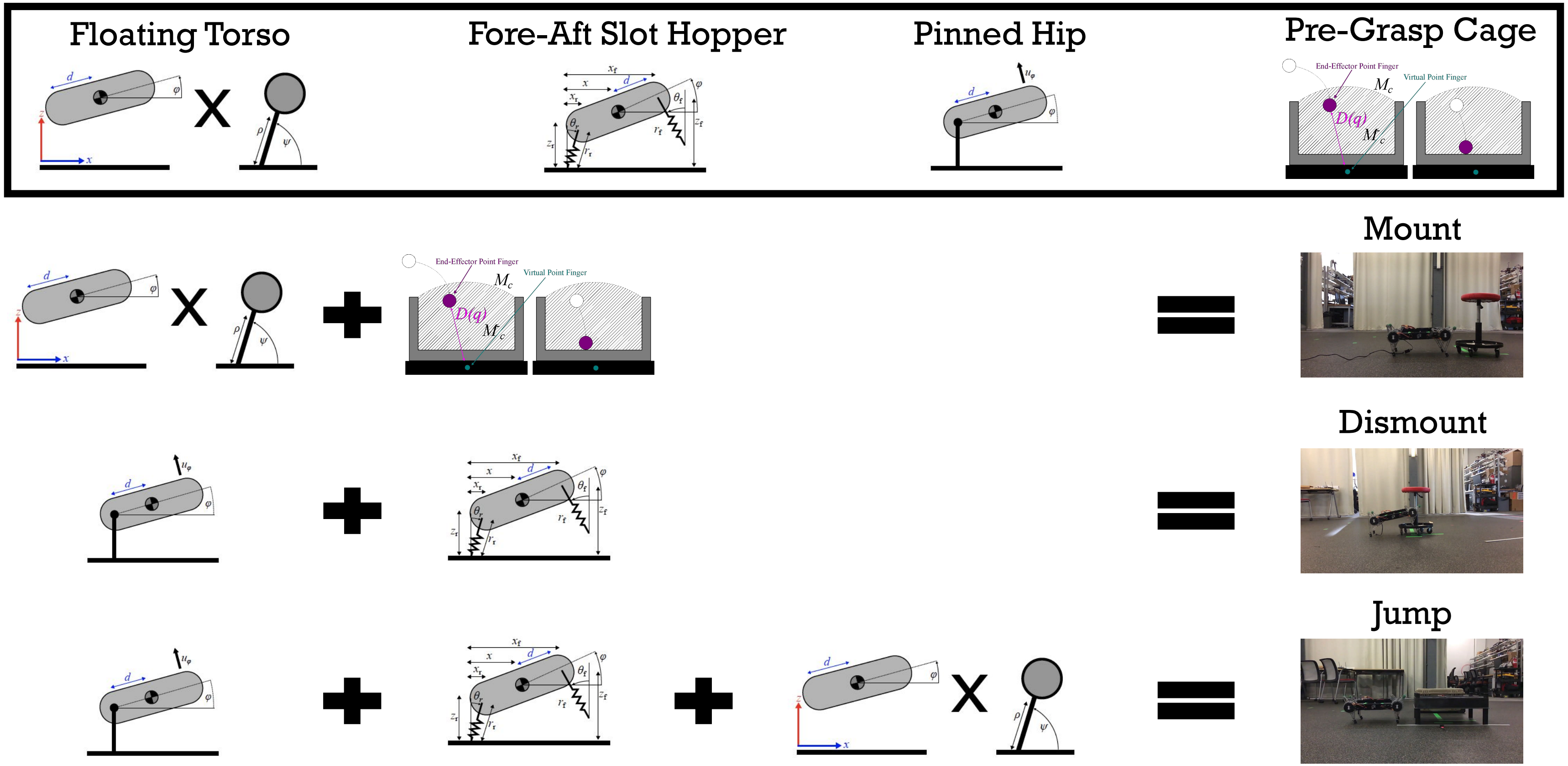






[R. Burridge, et al, “Sequential Composition of Dynamically Dexterous Robot Behaviors,” IJRR 1999.]
[A. De, et al, “Parallel Composition of Templates for Tail-Energized Hopping” ICRA 2015.]
[A. De, et al, “A Universal Template for Pitch Steady Behaviors in Planar Floating-Torso Locomotion Models,” In Prep, 2020.]
[V. Vasilopoulos, et al, “Sensor-Based Reactive Execution of Symbolic Rearrangement Plans by a Legged Mobile Manipulator” IROS 2018.]
[T. Topping, et al, “Quasi-Static and Dynamic Mismatch for Door Opening and Stair Climbing” ICRA 2017.]

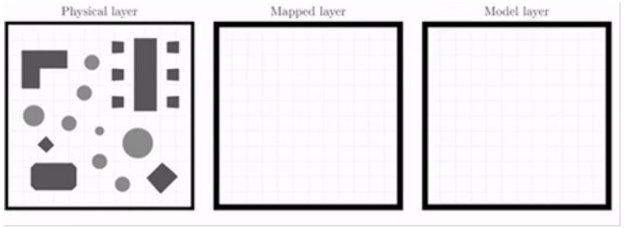
Emerging Calculus of Behaviors



[Topping et al., ISRR 2019]

Primary Constructions for Hybrid Systems (with P. Gustafson, D. Koditschek, P. Stiller)

Transformations



[Vasilopoulos, Koditschek, 2018]

A **hybrid semiconjugacy** $\alpha\colon H \to K$ is:

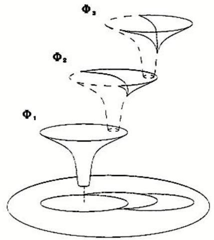
(1) a graph morphism $\alpha\colon G(H) \to G(K)$

(2) maps of active sets $\alpha_v\colon I_v^H \to I_v^K$

$$\begin{array}{ccc} F_v^H & \xrightarrow{X_v^H} & TF_v^H \\ \downarrow{\scriptstyle \alpha_v} & & \downarrow{\scriptstyle T\alpha_v} \\ F_v^K & \xrightarrow[X_v^K]{} & TF_v^K \end{array} \qquad \begin{array}{ccc} Z_e^H & \xrightarrow{r_e^H} & I_{\mathfrak{t}(e)}^H \\ \downarrow{\scriptstyle \alpha_{\mathfrak{s}(e)}|Z_e^H} & & \downarrow{\scriptstyle \alpha_{\mathfrak{t}(e)}} \\ Z_{\alpha(e)}^K & \xrightarrow[r_{\alpha(e)}^K]{} & I_{\mathfrak{t}(\alpha(e))}^K \end{array}$$

Hybrid semiconjugacies

Sequential Composition



[Burridge, Rizzi, Koditschek, 1999]

A **directed** system $H_{\mathrm{i}} \overset{H}{\rightsquigarrow} H_{\mathrm{f}}$ consists of

$$\begin{array}{ccc} & H & \\ {\scriptstyle \eta_{\mathrm{i}}}\nearrow & & \nwarrow{\scriptstyle \eta_{\mathrm{f}}} \\ H_{\mathrm{i}} & & H_{\mathrm{f}} \end{array}$$

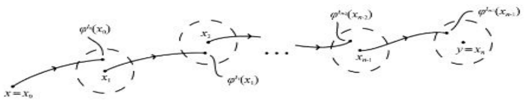
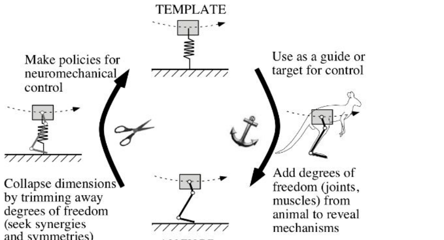


Image source: Alongi and Nelson, *Recurrence and Topology*. AMS, 2007.

Directed systems

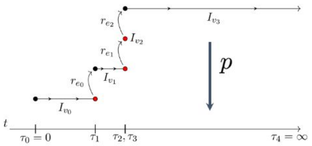
Templates & Anchors



[Koditschek, Full, 1999]

$$\begin{array}{ccc} & S & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle i} \\ T & & A \end{array}$$

$$\begin{array}{ccc} \sigma & \xrightarrow{\widetilde{\chi}} & S \\ \downarrow{\scriptstyle \xi} & \lrcorner & \downarrow{\scriptstyle p} \\ \tau & \xrightarrow[\chi]{} & H \end{array}$$

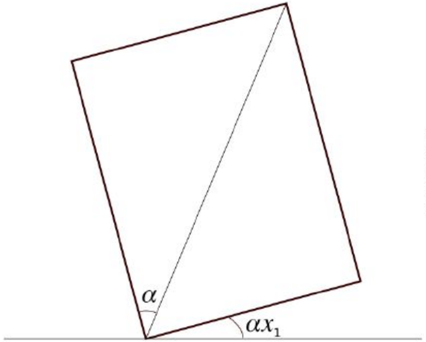


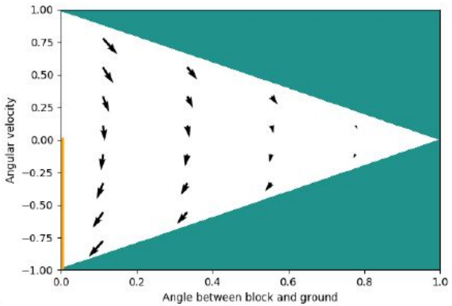
Hybrid subdivisions

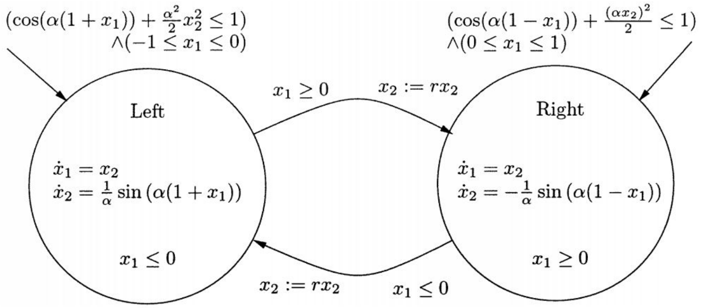
[C., Gustafson, Koditschek, Stiller, Formal Composition of Hybrid Systems, TAC, 2020]

A **hybrid system** H consists of

1. a directed graph* $G = (V, E, \mathfrak{s}, \mathfrak{t})$;
2. for each **continuous mode** $v \in V$,
   - an **ambient manifold** M_v
   - a vector field X_v on M_v
   - an **active set** $I_v \subset M_v$
   - a **flow set** $F_v \subset I_v$
3. for each **reset** $e \in E$, a **guard set** $Z_e \subset I_{\mathfrak{s}(e)}$ and an associated **reset map** $r_e \colon Z_e \to I_{\mathfrak{t}(e)}$.







[Image: Lygeros et al., "Dynamical properties of hybrid automata.," 2003].

* We work here in (a cat. equiv. to) the category of directed reflexive graphs.

A **hybrid semiconjugacy** $\alpha\colon H \to K$ is:

(1) a graph morphism $\alpha\colon G(H) \to G(K)$

(2) maps of active sets $\alpha_v\colon I_v^H \to I_v^K$

Notes:

- Conjugacy, submersions, embeddings straightforward
- Technical tools necessary to allow for practically-relevant subsets of manifolds (smooth sets)

$$\mathrm{Hom}_{\mathbf{S}}(A, B) = \operatorname*{colim}_{U \supseteq A} \{f \in \mathrm{Hom}_{\mathbf{M}}(U, M_A) \mid f(A) \subset B\}$$

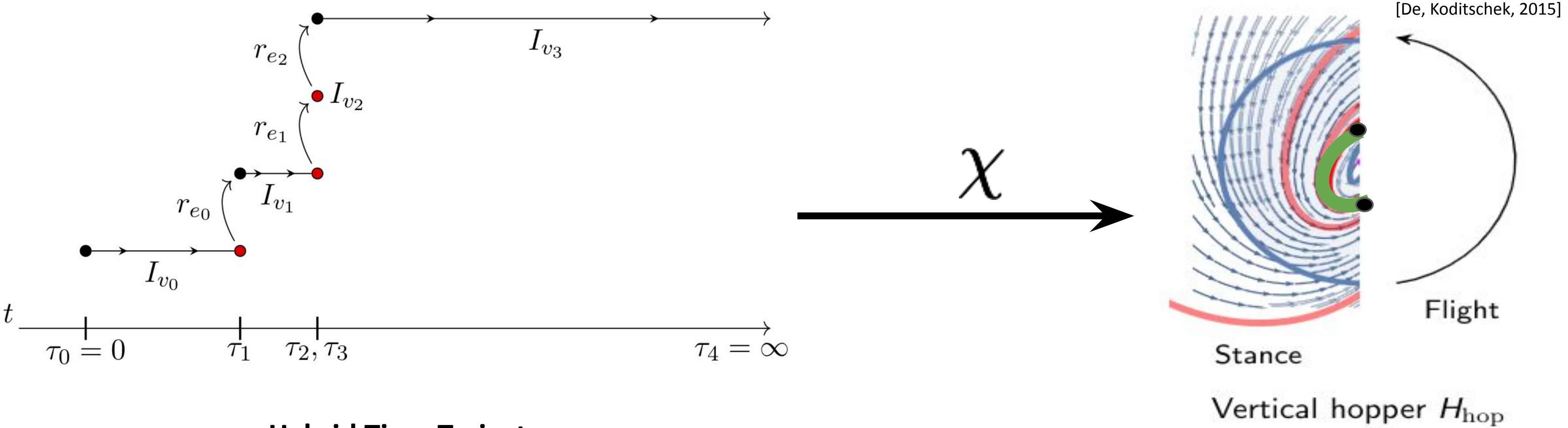
$$\begin{array}{ccc} F_v^H & \xrightarrow{X_v^H} & TF_v^H \\ \downarrow{\scriptstyle \alpha_v} & & \downarrow{\scriptstyle T\alpha_v} \\ F_v^K & \xrightarrow[X_v^K]{} & TF_v^K \end{array}$$

Flow Compatibility Condition

$$\begin{array}{ccc} Z_e^H & \xrightarrow{r_e^H} & I_{\mathfrak{t}(e)}^H \\ \downarrow{\scriptstyle \alpha_{\mathfrak{s}(e)}|_{Z_e^H}} & & \downarrow{\scriptstyle \alpha_{\mathfrak{t}(e)}} \\ Z_{\alpha(e)}^K & \xrightarrow[r_{\alpha(e)}^K]{} & I_{\mathfrak{t}(\alpha(e))}^K \end{array}$$

Reset Compatibility Condition

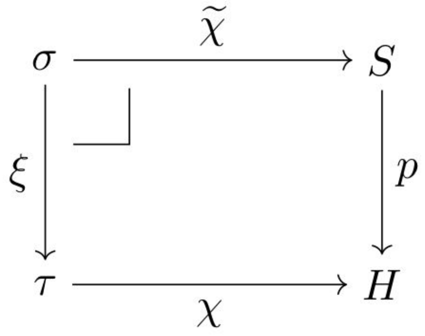
Hybrid executions as semiconjugacies



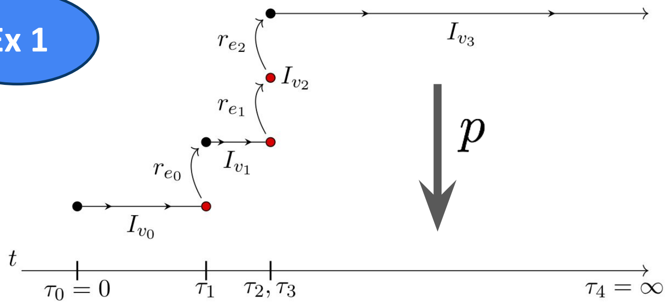
Hybrid Time Trajectory

Theorem (Pappas, Lerman, CGSK)
Semiconjugacies from hybrid time trajectories encode all executions of the codomain system.

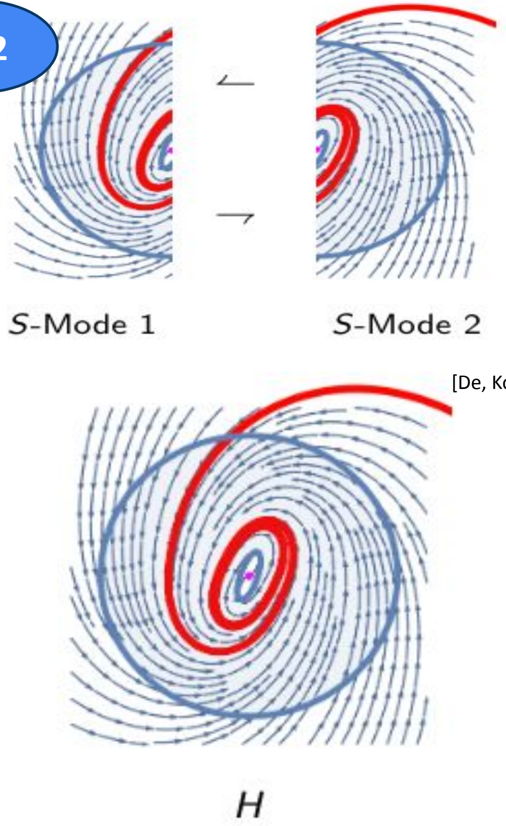
A **subdivision** is a hybrid submersion where



Ex 1



Ex 2

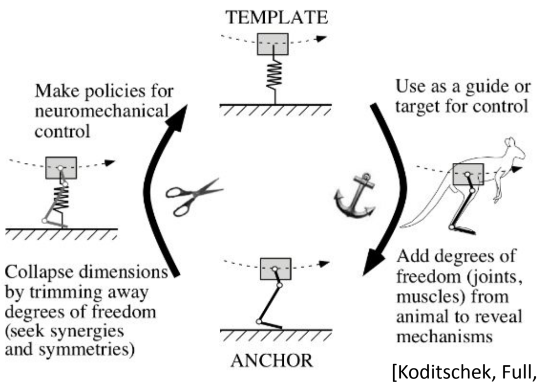


[De, Koditschek, 2015]

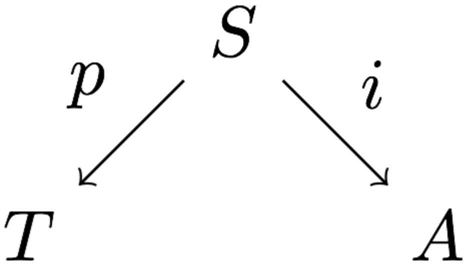
Slicing a Continuous Systems

A **template-anchor pair** is a span $T \overset{p}{\leftarrow} S \overset{i}{\rightarrow} A$ such that

- p is a hybrid subdivision;
- i is a hybrid embedding;
- $i(S)$ is attracting in A.

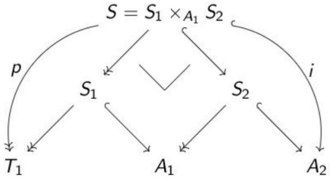


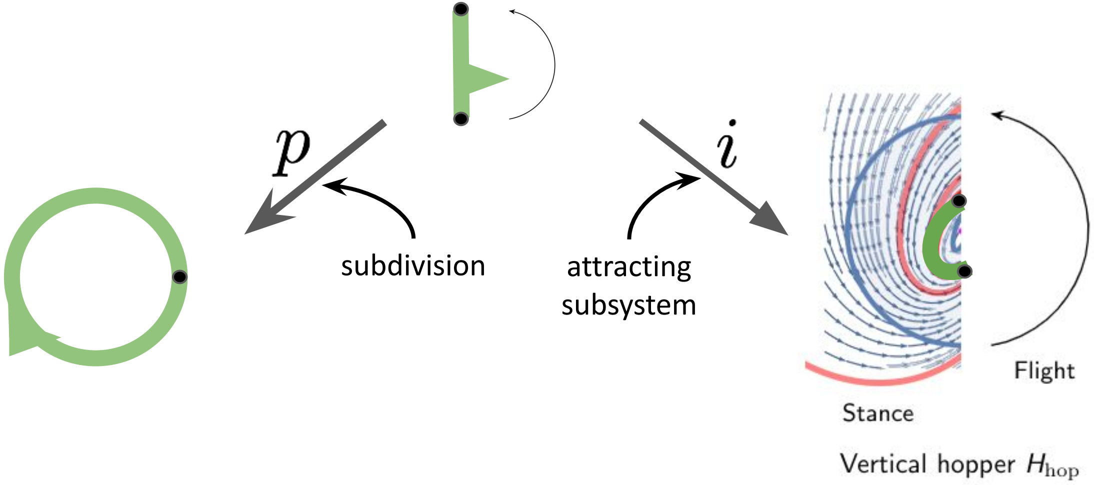
[Koditschek, Full, 1999]



Template-anchor relationship as a span

Theorem (CGKS). Template-anchor pairs are weakly associatively composable.





Example: hybrid limit cycle in grounded vertical hopper

[De, Koditschek, 2015]

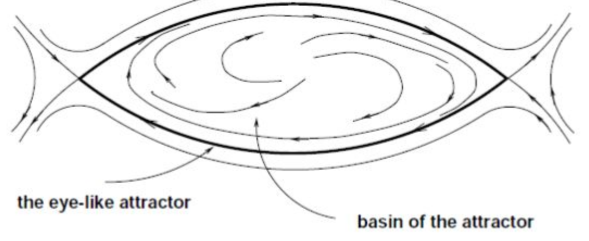
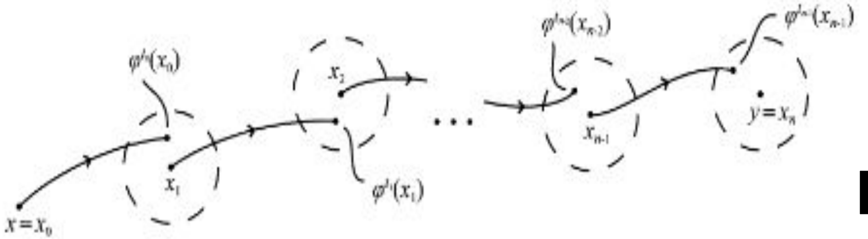


FIGURE 1. The Bowen example.

[Baladi, Bonatti, Bernard, 1999]

Conley's "Fundamental Theorem"

Theorem (Norton): *Every flow on a compact metric space decomposes into a chain-recurrent part and a gradient-like part.*



Classical (ε, T)-chains

Image source: Alongi and Nelson, *Recurrence and Topology*. AMS, 2007.

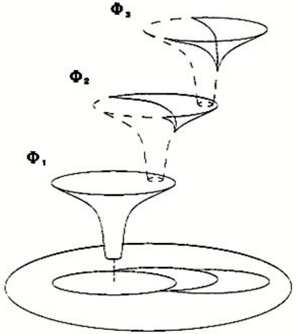
A **hybrid (ε, T)-chain** is roughly an execution with allowable ε-jumps

- after each reset
- after flowing for > T time

Theorem (Gustafson, Kvalheim, Koditschek, 2019)

Every hybrid system with compact state space satisfying a guard-set contraction property decomposes into a chain-recurrent part and a gradient-like part.

A **directed** system $H_{\mathbf{i}} \overset{H}{\rightsquigarrow} H_{\mathbf{f}}$ consists of



$$\begin{array}{ccc} & H & \\ \eta_{\mathrm{i}} \nearrow & & \nwarrow \eta_{\mathrm{f}} \\ H_{\mathrm{i}} & & H_{\mathrm{f}} \end{array}$$

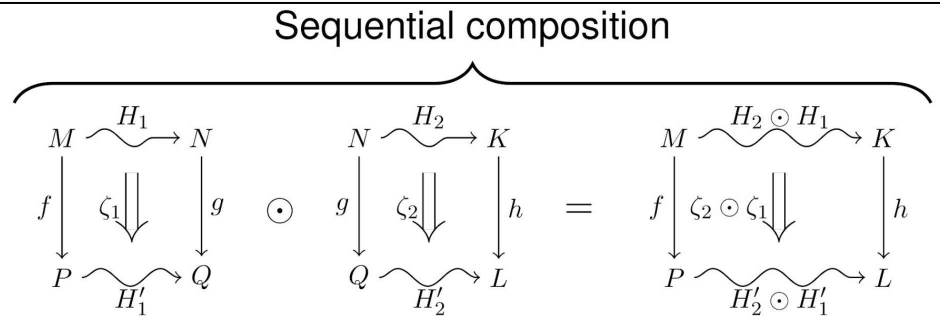
Initial subsystem — Final subsystem

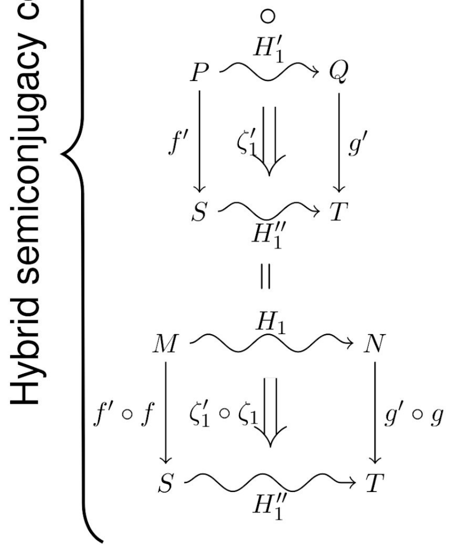
where $H_{\mathbf{f}}$ is an (ε, T)-sink in H

Key observations

1. State spaces not sufficient interfaces
   - Initial and final systems include dynamics
2. Standard executions do not compose well
   - (ε, T)-chains do compose!
   - Generalizes topological or measure theoretic approaches
3. “Same” system could have many applications
   - Different initial or final subsystems
   - Specifying new interface into system

Structural Results ⟶ Double Category of Hybrid Systems

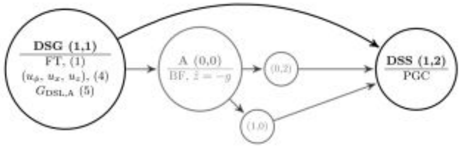


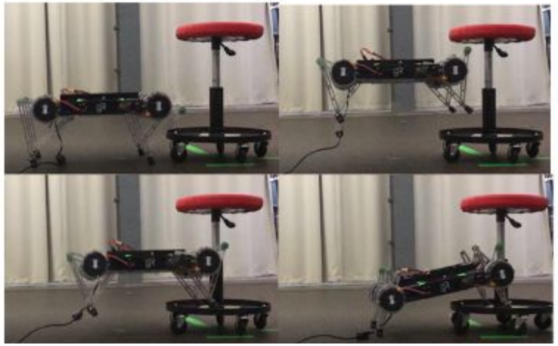


Category $\mathbf{H}$:

- objects are hybrid systems
- vertical category encodes hybrid semiconjugacy
- horizontal category encodes sequential composition
- has products and fiber products along hybrid submersions
- fibered over category of graphs
- framed bicategory if we allow reset relations
- associative template composition via spans

Theorem (CGKS)
Hybrid systems, directed systems and hybrid semiconjugacies form a cartesian and cocartesian double category .





[Example: Topping, Vasilopoulos, De, Koditschek, 2019]

Simple Type Theory for Mobile Manipulation

Type	Steady-state behavior
Free	no navigational goals
Object	stay near the object
Target_x	maintain position at $x \in X$
Nest	inhabit the nest

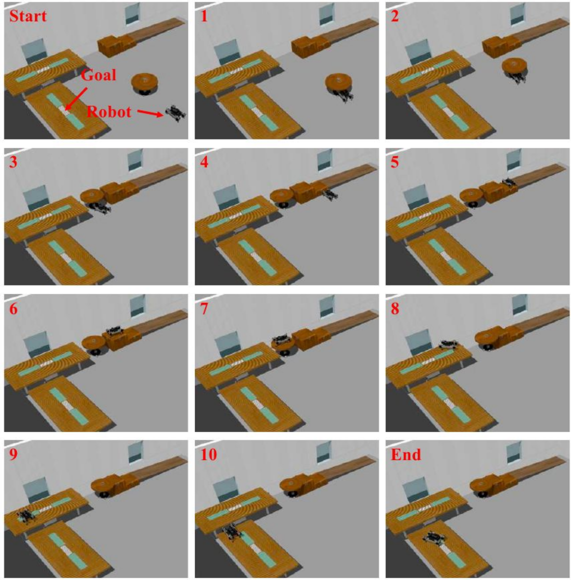
"Free type theory" on robot behaviors

Operational Semantics

Type	Continuous system (M, X)
Free	$(\mathbb{R}^2, 0)$
Object	$\left(B_\varepsilon(\Delta), X_d\vert_{B_\varepsilon(\Delta)}\right)$
Target_x	$\left(B_\varepsilon(x), X_t\vert_{B_\varepsilon(x)}\right)$
Nest	$\left(B_\varepsilon(r^*), X_n\vert_{B_\varepsilon(r^*)}\right)$

Morphism	Reactive hybrid system
$A \to \text{Object}$	$F_{\text{nav}}(A) \to (\mathbb{R}^2, X_d) \to (B_\epsilon(\Delta), X_d\vert_{B_\varepsilon(\Delta)})$
$A \to \text{Target}_x$	$F_{\text{nav}}(A) \to (\mathbb{R}^2, X_t) \to (B_\epsilon(x), X_t\vert_{B_\varepsilon(x)})$
$A \to \text{Nest}$	$F_{\text{nav}}(A) \to (\mathbb{R}^2, X_n) \to (B_\epsilon(r^*), X_n\vert_{B_\varepsilon(r^*)})$
$A \to \text{Free}$	$F_{\text{nav}}(A) \to (\mathbb{R}^2, 0)$

[Gustafson, C., Koditschek, Hybrid dynamical type theories for navigation, 2021]



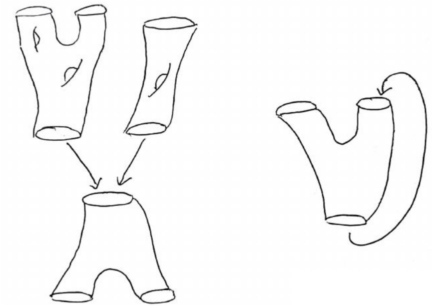
[Image:Vasilopoulos, 2021]

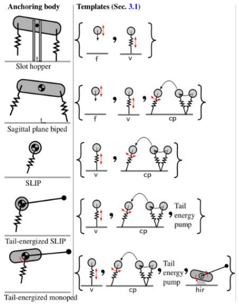
Implementation: https://github.com/PaulGustafson/HybSys

Choosing the right abstractions and simplifications

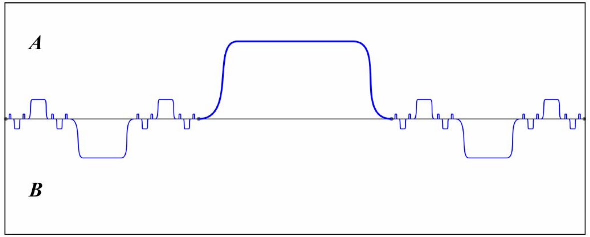
Examples:

1. More abstract and elegant formulations meet grounded approaches
    - Diffeological spaces, Frölicher spaces (exponential objects)
    - Manifolds with (generalized) corners (flows on products and subsets of smooth manifolds)
2. Simplifications
    - Not useful for us: gradient systems on surfaces, Morse systems, cobordisms, directed spaces
    - Still useful for us: only trivially coupled systems, no zeno behavior, no differential inclusions, restricting to nonblocking, deterministic systems





[De, Koditschek, 2018]



[Guralnik, 2020]

Concluding thoughts

1. Theorems of the form “* can be expressed as…” are not worth much in applications unless accompanied by meaningful machinery

2. Not every problem requires formalization/axiomatization

3. Working with practitioners continually is critical to making ***useful*** simplifying assumptions, abstractions, and interfaces