Higher categories in $\mathbb{C}\mathrm{at}^\sharp$

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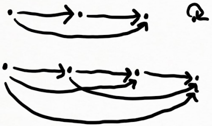
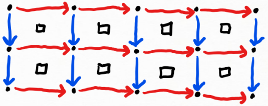
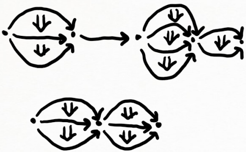
Topos Institute Colloquium

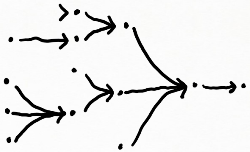
Categories with Different Cell Shapes

- Categories – dots, arrows
- 2-Categories – dots, arrows, globular 2-cells
- Double categories – dots, red/blue arrows, squares
- Multicategories – dots, n-to-1 arrows, $n \geq 0$



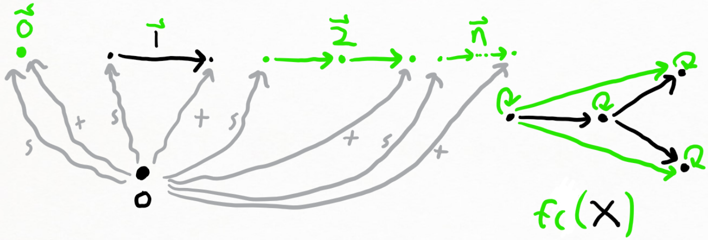






Familial Monads on Cell Diagrams

- G_1 is the category $0 \underset{t}{\overset{s}{\leftleftarrows}} 1$
- $G_1\text{-Set} = \mathrm{Set}^{G_1}$ is the category of graphs
- Categories are algebras for a monad fc on $\widehat{G_1}$
- $fc(X)_0 = X_0 = \mathrm{Hom}_{G_1\text{-Set}}(\cdot, X)$
  $fc(X)_1 = \{\text{paths in } X\} = \coprod_{n \geq 0} \mathrm{Hom}_{G_1\text{-Set}}(\cdot \to \overset{n}{\cdots} \to \cdot, X)$

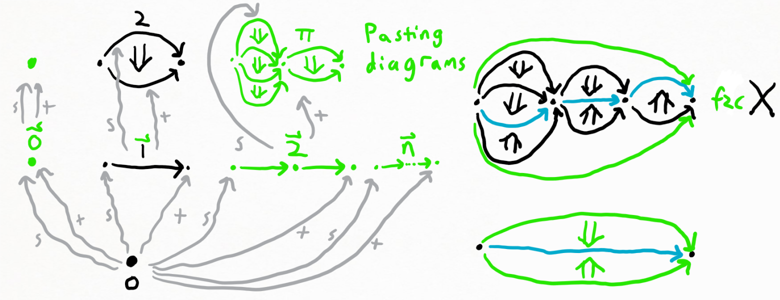


Familial Monads on Cell Diagrams

- G_2 is the category $0 \underset{t}{\overset{s}{\leftleftarrows}} 1 \underset{t}{\overset{s}{\leftleftarrows}} 2$
- G_2-Set is the category of 2-graphs
- 2-Categories are algebras for a monad $f2c$ on $\widehat{G_2}$
- $f2c(X)_0 = \mathrm{Hom}_{G_2\text{-Set}}(\cdot, X)$

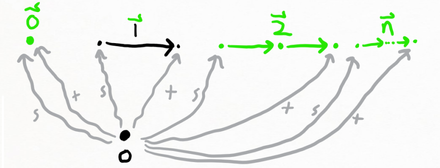
$$f2c(X)_1 = \coprod_{n \geq 0} \mathrm{Hom}_{G_2\text{-Set}}(\cdot \to \overset{n}{\cdots} \to \cdot, X)$$

$$f2c(X)_2 = \coprod_{\pi} \mathrm{Hom}_{G_2\text{-Set}}(\pi, X)$$



Familial Monads on Cell Diagrams

- The data of a familial functor $f : D\text{-Set} \to C\text{-Set}$ consists of:
  - A functor $f(1) : C \to \text{Set}$ (operations outputting a c-cell)
  - A functor $f[-] : \int S \to D\text{-Set}$ (arities of the operations)
- For c in C, X in D-Set, $f(X)_c = \coprod_{I \in f(1)_c} \text{Hom}_{D\text{-Set}}(f[I], X)$



Example: Free category monad on G_1-Set

- $fc(1)_0 = \{0\}$, $fc(1)_1 = \mathbb{N}$, $\quad fc[n] = \quad \cdot \to \overset{n}{\cdots} \to \cdot$
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- For c in C, X in D-Set, $f(X)_c = \coprod_{I \in f(1)_c} \text{Hom}_{D\text{-Set}}(f[I], X)$
- A monad (t, η, μ) on C-Set is familial if t is familial and η, μ are cartesian
- For 0 the empty category, a familial functor 0-Set $\to$ D-Set is just a single D-set.

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- Unit and multiplication on edges given by length 1 paths and path concatenation

Familial Monads in *Poly*

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In Poly-notation, $fc = \{0\}y^{fc[0]} + \{1\} \sum_{n \in fc(1)_1} y^{fc[n]}$.

- The monoidal category $(\mathrm{Poly}, y, \triangleleft)$ of polynomial endofunctors on Set consists of disjoint unions of representables y^A
- Categories agree with $\triangleleft$ -comonoids in Poly (Ahman-Uustalu)
- Bicomodules $C \xleftarrow{\;p\;} D$ in Poly agree with "prafunctors," aka familial functors C-Set $\leftarrow$ D-Set (Garner)
- Bicomodules $D \xleftarrow{\;X\;} 0$ are D-sets, and the composite $p \triangleleft_D X$ of bicomodules is the C-set $p(X)$

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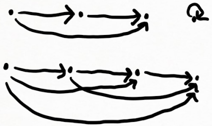
- Bicomodules $D \xleftarrow{X} 0$ are D-sets, and the composite $p \triangleleft_D X$ of bicomodules is the C-set $p(X)$
- $\mathbb{C}\mathrm{at}^\sharp$ is the bicategory of categories, prafunctors, and transformations
- A familial monad is a bicomodule $C \xleftarrow{t} C$, written

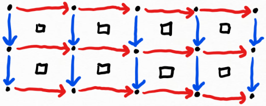
$$t = \sum_{c \in \mathrm{Ob}(C)} \sum_{I \in t(1)_c} y^{t[I]},$$

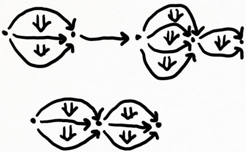
with cartesian transformations $\mathrm{id}_C \to t$ and $t \triangleleft_C t \to t$

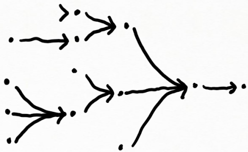
Nerves of Higher Categories

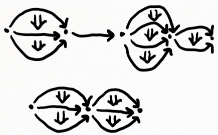
- Categories $\rightarrow \widehat{\Delta}$ simplicial sets
- 2-Categories $\rightarrow \widehat{\Theta_2}$ Θ_2-sets
- Double categories $\rightarrow \widehat{\Delta \times \Delta}$ bisimplicial sets
- Multicategories $\rightarrow \widehat{\Omega}$ dendroidal sets

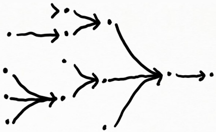




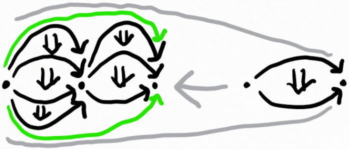


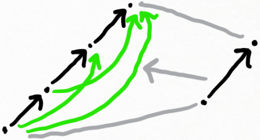




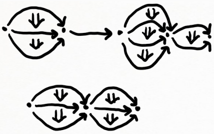


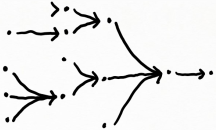
- $$t = \sum_{c \in \mathrm{Ob}(C)} \sum_{I \in t(1)_c} y^{t[I]}$$
  with cartesian transformations $\mathrm{id}_C \to t$ and $t \triangleleft_C t \to t$
- (Weber '07) There is a fully faithful functor t-alg $\to \Theta_t^{op}$-Set for a category Θ_t with objects $\coprod_{c \in \mathrm{Ob}(C)} t(1)_c$ and
  $$\mathrm{Hom}(I, J) = \mathrm{Hom}_{t\text{-alg}}(t(t[I]), t(t[J]))$$
- Morphisms include "cocompositions" $y^c \to t(y^c) \to t(t[I])$





Nerves of Higher Categories

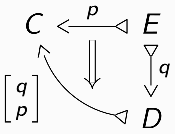


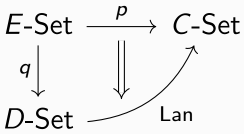


- (Weber '07) There is a fully faithful functor t-alg $\to \Theta_t^{op}$-Set for a category Θ_t with objects $\coprod_{c \in \mathrm{Ob}(C)} t(1)_c$ and

$$\mathrm{Hom}(I, J) = \mathrm{Hom}_{t\text{-alg}}(t(t[I]), t(t[J]))$$

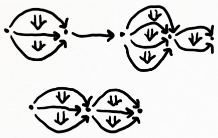
- For bicomodules p, q as below,

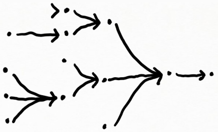




there is a bicomodule $\begin{bmatrix} q \\ p \end{bmatrix} := \sum_{c \in \mathrm{Ob}(C)} \sum_{I \in p(1)_c} y^{q(p[I])}$

Nerves of Higher Categories





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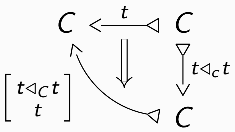
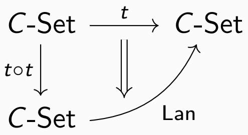
- For the bicomodule t,





there is a bicomodule $\begin{bmatrix} t \triangleleft_C t \\ t \end{bmatrix} = \sum_{c \in \mathrm{Ob}(C)} \sum_{I \in t(1)_c} y^{t(t(t[I]))}$

Nerves of Higher Categories

- (Weber '07) There is a fully faithful functor $t\text{-alg} \to \Theta_t^{op}\text{-Set}$ for a category Θ_t with objects $\coprod_{c \in \mathrm{Ob}(C)} t(1)_c$ and

$$\mathrm{Hom}(I, J) = \mathrm{Hom}_{t\text{-alg}}(t(t[I]), t(t[J]))$$

- For the bicomodule t,

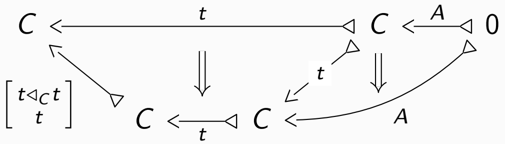
there is a bicomodule $\begin{bmatrix} t \triangleleft_C t \\ t \end{bmatrix} = \sum_{c \in \mathrm{Ob}(C)} \sum_{I \in t(1)_c} y^{t(t(t[I]))}$

- $\begin{bmatrix} t \triangleleft_C t \\ t \end{bmatrix}$ is a comonoid corresponding to the category Θ_t^{op}, as

$$t(t(t[I])) = \sum_{c \in \mathrm{Ob}(C)} \sum_{J \in p(1)_c} t(t[I])^{t[J]} \cong \sum \mathrm{Hom}_{t\text{-alg}}(t(t[J]), t(t[I]))$$

Nerves of Higher Categories

- (Weber '07) There is a fully faithful functor $t\text{-alg} \to \Theta_t^{op}\text{-Set}$ for a category Θ_t with objects $\coprod_{c \in \mathrm{Ob}(C)} t(1)_c$ and
$$\mathrm{Hom}(I, J) = \mathrm{Hom}_{t\text{-alg}}(t(t[I]), t(t[J]))$$
- $\begin{bmatrix} t \triangleleft_C t \\ t \end{bmatrix} = \sum_{c \in \mathrm{Ob}(C)} \sum_{I \in t(1)_c} y^{t(t(t[I]))}$ is a comonoid corresponding to the category Θ_t^{op}
- A t-algebra can be modeled as a bicomodule $C \overset{A}{\Leftarrow\!\!\triangleleft} 0$ with a transformation $t \triangleleft_c A \to A$
- (Lynch-S.-Spivak) The nerve of an algebra A is $t(A)$, which is a Θ_t^{op}-set as $t \triangleleft_C A$ has a Θ_t^{op}-coalgebra structure:



References

- Owen Lynch, Brandon T. Shapiro, David I. Spivak, "All Concepts are $\mathbb{C}\mathrm{at}^\sharp$." arXiv:2305.02571
- Brandon T. Shapiro, Thesis "Shape Independent Category Theory." pi.math.cornell.edu/∼bts82/research
- David I. Spivak, "Functorial Aggregation." arXiv:2111.10968
- Mark Weber, "Familial 2-functors and parametric right adjoints." TAC18-22

Thanks!