Monoidal bicategories, differential linear logic, and analytic functors

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(jww M. Fiore & M. Hyland)

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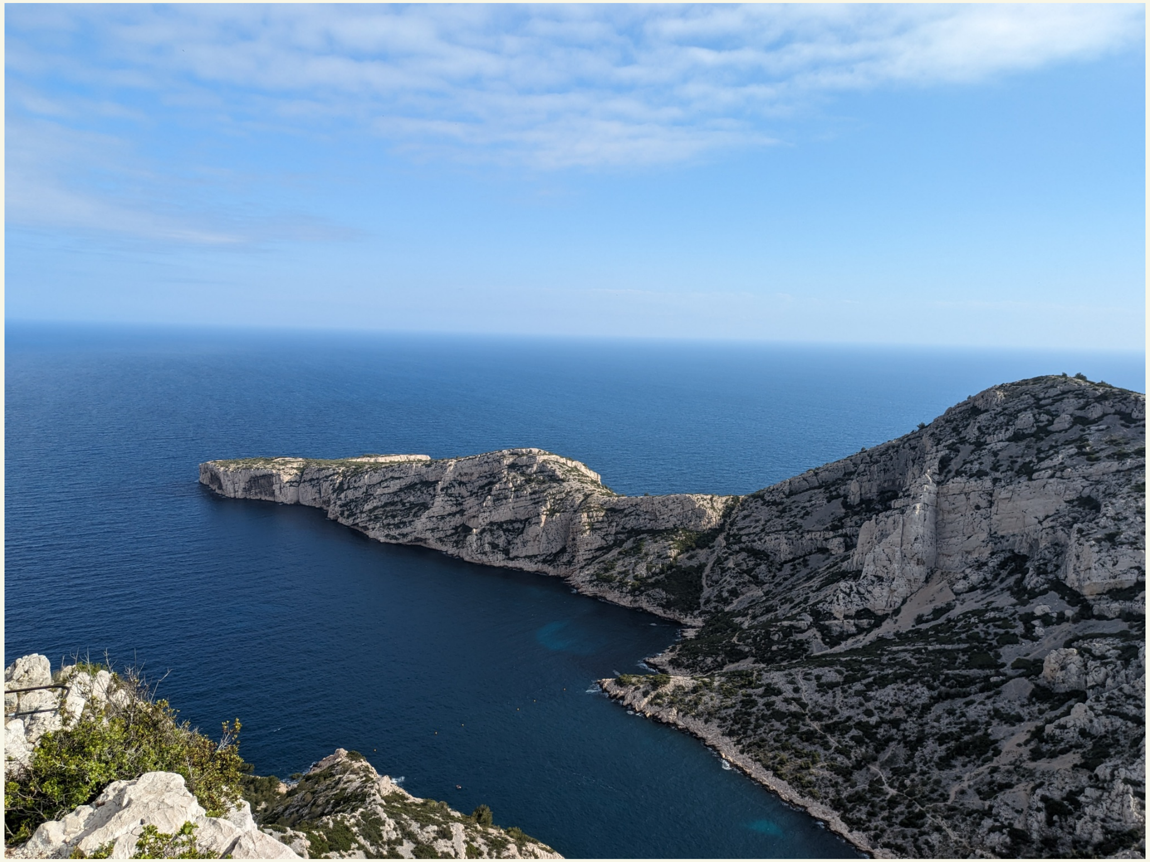
Last week...



Luminy (CIRM)

Diff. λ-calculus and diff. linear logic, 20 years later

https://conferences.cirm-math.fr/2980.html

Plan

1. Monoidal bicategories

2. Differential linear logic

3. Analytic functors

Motivation

Logic	**Bi Category theory**
Formula A	Object A
Sequent $\Gamma \vdash A$	Map $\Gamma \xrightarrow{a} A$
	2-cells $\Gamma \underset{a'}{\overset{a}{\Downarrow}} A$
Cut elimination rules, e.g.	Equations between maps

$$\dfrac{\dfrac{\Gamma \vdash A}{\Gamma \vdash !A}\,\bar{d} \qquad \dfrac{A \vdash B}{!A \vdash B}\,d}{\Gamma \vdash B}\,\text{cut}$$

$$\rightsquigarrow \qquad \dfrac{\Gamma \vdash A \qquad A \vdash B}{\Gamma \vdash B}\,\text{cut}$$

$$\begin{tikzcd}
\Gamma \arrow[r, "a"] \arrow[ddr, bend right, "a"'] & A \arrow[r, "\bar{d}"] \arrow[dr, bend right, "1_A"'] \arrow[dd, phantom, "\Swarrow"] & !A \arrow[d, "d"] \arrow[dl, phantom, "\Swarrow"] \\
 & & A \arrow[d, "b"] \\
 & A \arrow[r, bend right, "b"'] & B
\end{tikzcd}$$

equations between rewritings.

equations between 2-cells

1. Monoidal bicategories

Models of differential linear logic in 1d : based on

- symmetric monoidal categories
- linear exponential comonad

$\Rightarrow$ we are interested in 2d versions.

Definition. A bicategory K has

- objects: $A, B, C, \ldots$
- hom-categories $K(A,B)$ — $f : A \longrightarrow B$; $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\phi"] \end{tikzcd}$
- composition functors
- identity maps $1_A : A \longrightarrow A$
- iso's $\begin{tikzcd} A \arrow[r, bend left, "h\cdot(g\cdot f)"{name=U}] \arrow[r, bend right, "(h\cdot g)\cdot f"'{name=D}] & C \arrow[from=U, to=D, Rightarrow] \end{tikzcd}$, $\begin{tikzcd} A \arrow[r, bend left, "f\cdot 1_A"{name=U}] \arrow[r, bend right, "f"'{name=D}] & B \arrow[from=U, to=D, Rightarrow] \end{tikzcd}$, $\begin{tikzcd} A \arrow[r, bend left, "1_B\cdot f"{name=U}] \arrow[r, bend right, "f"'{name=D}] & B \arrow[from=U, to=D, Rightarrow] \end{tikzcd}$

subject to axioms.

Example **Prof** has

- objects : small categories $A, B, C, \ldots$
- $\mathbf{Prof}[A, B] = \mathrm{CAT}[B^{op} \times A, \underline{\mathrm{Set}}]$

$F : B^{op} \times A \longrightarrow \mathrm{Set}$
$(b, a) \longmapsto F(b, a)$

- composition : $A \xrightarrow{F} B \xrightarrow{G} C$

$$(G \circ F)(c, a) = \int^{b \in B} G(c, b) \times F(b, a)$$

- identity $1_A(a', a) = A[a', a]$

Example Monoidal category $(\mathcal{C}, \otimes, I)$

Strictification Thm (Mac Lane & Paré)

Every bicategory is equivalent to a 2-category.

Definition A monoidal bicategory is a one-object tricategory.

Strictification Thm (Gordon, Power, Street ; Gurski)

Every monoidal bicategory is equivalent to a Gray monoid

K with

$K \times K \xrightarrow{\otimes} K$ semistrict , $I \in K$

s.t.

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C , \quad A \otimes I = A = I \otimes A .$$

Recall: in ~~1d~~ 2d, for $(\mathbb{C}, \otimes, I)$ Gray monoid we have

- **braiding**: $A \otimes B \xrightarrow{r_{A,B}} B \otimes A$ s.t.

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[r, "r \otimes 1"] \arrow[dr, bend right, "r"'] & B \otimes A \otimes C \arrow[d, "1 \otimes r"] \arrow[dl, phantom, "\Downarrow"] \\
& B \otimes C \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \otimes C \arrow[r, "1 \otimes r"] \arrow[dr, bend right, "r"'] & A \otimes C \otimes B \arrow[d, "r \otimes 1"] \arrow[dl, phantom, "\Downarrow"] \\
& C \otimes A \otimes B
\end{tikzcd}$$

- ~~symmetry~~ sylleptic: if

$$\begin{tikzcd}
A \otimes B \arrow[r, "r"] \arrow[dr, bend right, "1"'] & B \otimes A \arrow[d, "r"] \arrow[dl, phantom, "\Downarrow"] \\
& A \otimes B
\end{tikzcd}$$

- **symmetry**: if

$$\begin{tikzcd}
A \otimes B \arrow[r, "r"] \arrow[dr, bend right] & B \otimes A \arrow[d, "r"] \arrow[dl, phantom, "\cong"] \\
& A \otimes B \arrow[r, "r"] & B \otimes A
\end{tikzcd}$$

$$= \quad \begin{tikzcd}
A \otimes B \arrow[r, "r"] & B \otimes A \arrow[d, "r"'] \arrow[dr, bend left, "1"] \arrow[dr, phantom, "\cong"] \\
& A \otimes B \arrow[r, "r"'] & B \otimes A
\end{tikzcd}$$

Strictification Theorem (Gurski & Osorno)

Every symmetric monoidal bicategory is equivalent to a symmetric Gray monoid.

Fact There is a theory of monoidal bicategories (Kapranov-Voevodsky, Baez & Neuchl, ... , Day & Street, ...)

Let $(K, \otimes, I)$ be a sym. Gray monoid.

cf Hyland & Schalk

Definition A linear exponential pseudocomonad on K is

- a sym. lax monoidal pseudocomonad

$$K \xrightarrow{!} K \quad , \quad !A \xrightarrow{p_A} !!A \quad , \quad !A \xrightarrow{d_A} A \quad ;$$

- every pseudocoalgebra $A \in K^!$ is a sym. pseudocomonoid

$$A \xrightarrow{n_A} A \otimes A \, , \qquad A \xrightarrow{e_A} I \quad ;$$

- $(n_A)_{A \in K}$, $(e_A)_{A \in K}$ are braided monoidal psd-nat. transformations ;

- assoc. , unit. , symm. constraints of the sym. pseudocomonoid are monoidal modifications.

Theorem (Miranda) $(!, p, d)$ sym. lax monoidal pseudocomonad on $(K, \otimes, I) \Rightarrow (K^!, \otimes, I)$ sym. monoidal bicategory.

Theorem $(!, p, d)$ linear exponential pseudocomonad on $(K, \otimes, I)$. Then:

- $(K^!, \otimes, I)$ is cartesian.
- if K has products $(K, \&, T) \xrightarrow{!} (K, \otimes, I)$ sylleptic strong monoidal (Seely equivalences)

Example

Write pseudocomonad in Kleisli form :

$$\frac{!A \xrightarrow{f} B}{!A \xrightarrow{f^*} !B}$$

$$\begin{tikzcd} !A \arrow[r, bend left, "d_A^*"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow"] & !A \end{tikzcd} \qquad \begin{tikzcd} & !B \arrow[dr, "g^*"] \arrow[d, phantom, "\Downarrow"] & \\ !A \arrow[ur, "f^*"] \arrow[rr, bend right, "(gf^*)^*"'] & {} & !C \end{tikzcd} \qquad \begin{tikzcd} & !B \arrow[dr, "d_B"] \arrow[d, phantom, "\Downarrow"] & \\ !A \arrow[ur, "f^*"] \arrow[rr, bend right, "f"'] & {} & B \end{tikzcd}$$

a coherence axiom :

$$\begin{tikzcd} & !B \arrow[dr, "d_B^*"] \arrow[d, phantom, "\Downarrow"] & \\ !A \arrow[ur, "f^*"] \arrow[rr, "(d_B f^*)^*" description] \arrow[rr, bend right=70, "f^*"'] & {} \arrow[d, phantom, "\Downarrow"] & !B \\ & {} & \end{tikzcd} \quad = \quad \begin{tikzcd} !A \arrow[r, "f^*"] \arrow[rr, bend right=60, "f^*"'] \arrow[rr, phantom, bend right=30, "\cong"] & !B \arrow[r, bend left, "d_B^*"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow"] & !B \end{tikzcd}$$

$$\dfrac{\dfrac{!A \xrightarrow{f} B}{!A \xrightarrow{f^*} !B} \qquad \dfrac{!B \xrightarrow{d_B} B}{!B \xrightarrow{d_B^*} !B}}{!A \xrightarrow{d_B^* f^*} !B}\ \color{red}{\text{cut}}$$

$\swarrow \qquad\qquad\qquad\qquad \searrow$

$$\dfrac{\dfrac{\dfrac{!A \xrightarrow{f} B}{!A \xrightarrow{f^*} !B} \qquad !B \xrightarrow{d_B} B}{!A \xrightarrow{d_B f^*} B}\ \color{red}{\text{cut}}}{!A \xrightarrow{(d_B f^*)^*} !B}$$

$$\dfrac{\dfrac{!A \xrightarrow{f} B}{!A \xrightarrow{f^*} !B} \qquad !B \xrightarrow{1} !B}{!A \xrightarrow{f^*} !B}\ \color{red}{\text{cut}}$$

$\searrow \qquad\qquad\qquad\qquad \swarrow\ \color{red}{\cong}$

$$\dfrac{!A \xrightarrow{f} B}{!A \xrightarrow{f^*} !B}$$

2. Differential linear logic

Let $(\mathcal{K}, \otimes, I)$ be a sym. Gray monoid, $(!, p, d)$

a linear exponential pseudocomonad on $\mathcal{K}$.

WANT A version of 'codereliction', ie

$$\bar{d}_A : A \longrightarrow !A$$

which allows, in good cases, to define derivatives

$$!A \xrightarrow{f} B \quad \rightsquigarrow \quad A \otimes !A \xrightarrow{df} B$$

Assume $d_A : \,!A \longrightarrow A$ has a left adjoint $\bar{d}_A : A \longrightarrow \,!A$

with unit

$$
\begin{tikzcd}
A \arrow[r, "\bar{d}_A"] \arrow[dr, bend right, "1_A"'] \arrow[dr, phantom, "\Rightarrow"] & !A \arrow[d, "d_A"] \\
 & A
\end{tikzcd}
$$

not nec. invertible !

<u>Theorem</u> Assume . Then there are 2-cells

(i)

$$
\begin{tikzcd}
A \arrow[rr, "\bar{d}"] \arrow[d, "\bar{d} \otimes \bar{w}"'] \arrow[drr, phantom, "\Rightarrow"] & & !A \arrow[d, "p"] \\
!A \otimes !A \arrow[r, "\bar{d}_! \otimes p"'] & !!A \otimes !!A \arrow[r, "\bar{c}_!"'] & !!A
\end{tikzcd}
$$

(Alternative chain rule)

(ii)

$$
\begin{tikzcd}
A \otimes !B \arrow[r, "\bar{d} \otimes 1"] \arrow[d, "1 \otimes \bar{d}"'] \arrow[dr, phantom, "\Rightarrow"] & !A \otimes !B \arrow[d] \\
A \otimes B \arrow[r, "\bar{d}"'] & !(A \otimes B)
\end{tikzcd}
$$

(Monoidal rule)

Definition A codereliction is a pseudonatural transformation with components $\bar{d}_A : A \longrightarrow\, !A$ adjoint to $d_A : !A \longrightarrow A$ such that the 2-cells η, σ, μ are invertible.

Note One way to address coherence.

See

- Blute, Cockett, Lemay, Seely, "Diff. categories revisited"
- Fiore, "Diff. structure in.."

3. Analytic functors

The bicategory <u>Prof</u>

- sym. monoidal : $A \otimes B =_{def} A \times B$
- biproducts : $A \oplus B =_{def} A + B$
- compact closed $A^{\perp} =_{def} A^{op}$
- internal hom : $A \multimap B = A^{op} \times B$

Theorem The free sym. strict monoidal category 2-monad on **Cat** extends to a linear exponential pseudocomonad on **Prof**.

$!A$ has

- objects : $\alpha = \langle a_1, \dots, a_n \rangle$
- maps : $(\sigma, f) : \langle a_1, \dots, a_n \rangle \longrightarrow \langle a_1', \dots, a_m' \rangle$

  where $\sigma \in \mathrm{Bij}(n,m)$, $f_i : a_i \longrightarrow a'_{\sigma(i)}$.

Definition Let $A, B \in \underline{\mathrm{Cat}}$. A categorical symmetric sequence $M : A \to B$ is a profunctor $M : {!A} \to B$.

Kleisli map

Given $M : A \to B$, define the analytic functor

$$\begin{array}{ccc} \mathrm{Psh}(A) & \xrightarrow{F_M} & \mathrm{Psh}(B) \\ X & \longmapsto & \displaystyle\int^{\alpha \in !A} M[-,\alpha] \times X^{\alpha} \end{array}$$

$f(x) = \sum f_n \frac{x^n}{n!}$

where $X^{\alpha} = X(a_1) \times \ldots \times X(a_n)$

Note For $A = B = 1$, get Joyal's analytic functors.

Theorem $(\underline{\mathrm{Prof}}, \otimes, I)$ with $(!, p, d)$ admit a codereliction operation $\bar{d}_A : A \longrightarrow\ !A$ given by

$$\begin{aligned} !A^{op} \times A &\longrightarrow \underline{\mathrm{Set}} \\ \alpha\ ,\ a &\longmapsto\ !A[\alpha, \langle a \rangle] \end{aligned}$$

$\Rightarrow$ We can now take the derivative of Kleisli maps = analytic functors

$$\frac{!A \xrightarrow{F} B \quad \text{in } \underline{\mathrm{Prof}}}{A \otimes !A \xrightarrow{dF} B \quad \text{in } \underline{\mathrm{Prof}}}$$

$$\frac{}{!A \xrightarrow{dF} (A \multimap B) \quad \text{in } \underline{\mathrm{Prof}}}$$

$$dF(b, (a, \alpha)) =_{def} F(b, \langle a \rangle \uplus \alpha)$$

$$\mathrm{Psh}(A) \longrightarrow \mathrm{Psh}(A \multimap B)$$

$$X \longmapsto \int^{\alpha \in !A} F(b, \langle a \rangle \uplus \alpha) \times X^{\alpha}$$

cf

$$f'(x) = \sum_n f_{n+1} \frac{x^n}{n!}$$

Note For $A = B = 1$, get Joyal's differentiation for analytic functors on Set.

Summary

The bicategory $\underline{\mathrm{Prof}}_!$

- ✓ is cartesian closed [FGHW], [GJ].
- ✓ has differentiation [FGH].
- ✓ has oplax monoidal structure [GGV].
- ✓ has fixpoints [Galal].
- ✓ coloured operads live naturally in $\mathrm{Prof}_!$ [GJ].