

Three Toposes for Probability & Randomness

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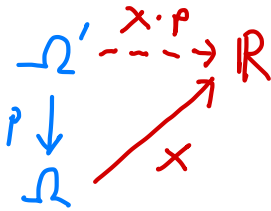
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Topos 1

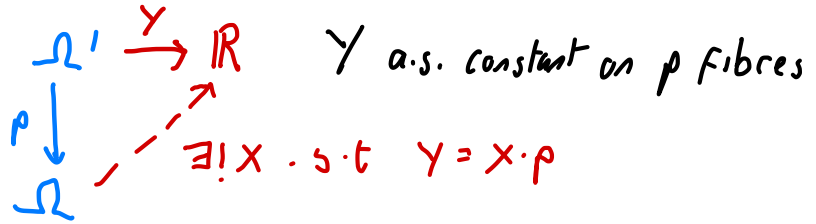
(Probability)

Random variables form a sheaf

Presheaf property



Sheaf property



(w.r.t. atomic Grothendieck topology)

The topos of probability sheaves

Base categories of 'nice' sample spaces

\mathcal{SIBIP} Standard Borel probability spaces + measure-preserving functions

\mathcal{SIBIP}_0 " " " " " " " mod 0

The topos

$$\underline{\mathcal{P}} := \text{Sh}_{\text{at}}(\mathcal{SIBIP}_0) \cong \text{Sh}_{\text{at}}(\mathcal{SIBIP})$$

(atomic
sheaves)

The sheaf $\underline{RV}(A)$

(of A -valued random variables)

For any standard Borel space A (with σ -algebra \mathcal{B}_A)

$$\begin{array}{ccc} \Omega & & \underline{RV}(A)(\Omega) := \text{measurable functions } \Omega \rightarrow A \text{ mod } 0 \\ \uparrow [\rho] & \mapsto & \downarrow [x] \mapsto [x \circ \rho] \\ \Omega' & & \underline{RV}(A)(\Omega') \end{array}$$

defines $\underline{RV}(A) : \mathfrak{SIBP}_0^{\text{op}} \rightarrow \underline{\text{Set}}$ that is a sheaf in $\underline{\mathcal{P}}$.

The RV functor

The mapping $A \mapsto \underline{RV}(A)$ defines a functor

$$\underline{RV}: \text{\$BS} \rightarrow \underline{\mathcal{P}}$$

Category of standard Borel spaces and measurable functions.

Properties:

- faithful
- preserves countable limits.

(The domain category can be expanded to universally measurable functions between universally measurable subsets of standard Borel spaces.)

The distribution functor

$$A \mapsto \{ \mu: \mathcal{B}_A \rightarrow [0,1] \mid \mu \text{ a probability measure} \}$$

defines a functor

$$\underline{D}: \mathcal{SBS} \rightarrow \underline{\text{Set}} \xrightarrow{\Delta} \underline{\mathcal{P}}$$

discrete presheaf

Properties:

- faithful
- taut

The law of a random variable

$$\mathbb{P}_A^X : \underline{RV}(A)(\Omega) \rightarrow \underline{D}(A)(\Omega)$$

$$X \mapsto (B \in \mathcal{B}_A \mapsto \mathbb{P}[X \in B])$$

the law of X

defines a natural transformation

$$\begin{array}{ccc} \underline{\mathcal{B}} & \xrightarrow{\underline{RV}} & \underline{\mathcal{P}} \\ \mathbb{P} \Downarrow & & \\ \underline{\mathcal{D}} & \xrightarrow{\underline{D}} & \underline{\mathcal{P}} \end{array}$$

- \mathbb{P} is taut

Internal definitions of probabilistic concepts

Using the internal logic of $\underline{\mathcal{P}}$, which is classical because atomic toposes are boolean.

For $X, Y: \underline{RV}(A)$

$$X \sim Y \quad :\Leftrightarrow \quad \mathbb{P}_X = \mathbb{P}_Y$$

abbreviation for $\mathbb{P}_A(X)$

Proposition $X =_{a.s.} Y \Leftrightarrow X = Y$!
(follows from tautness of \mathbb{P})

$$X =_{a.s.} Y \quad :\Leftrightarrow \quad \mathbb{P}_{(X,Y)}(\{(x,x) \mid x \in A\}) = 1$$

using product preservation

$\underline{RV}(A) \times \underline{RV}(A) \cong \underline{RV}(A \times A)$ to
consider (X, Y) as element of $\underline{RV}(A \times A)$

For $X: \underline{RV}(A), Y: \underline{RV}(B)$

$$X \perp\!\!\!\perp Y \quad :\Leftrightarrow \quad \forall S \in \Delta \mathcal{B}_A, \forall T \in \Delta \mathcal{B}_B. \mathbb{P}_{(X,Y)}(S \times T) = \mathbb{P}_X(S) \cdot \mathbb{P}_Y(T).$$

Two logical laws

Invariance principle

For any subsheaf $\phi \rightsquigarrow \underline{RV}(A)$,

$$\forall x, y : \underline{RV}(A) \quad x \sim y \wedge \phi(x) \rightarrow \phi(y)$$

Independence principle

$$\forall x : \underline{RV}(A), y : \underline{RV}(B) \quad \exists z : \underline{RV}(A) \quad z \sim x \wedge z \perp\!\!\!\perp y$$

Dependent choice

A topos with countable limits enjoys the principle of (internal countable) dependent choice (DC) if

every ω^{op} -diagram of epimorphisms

$$\dots \xrightarrow{e_4} X_4 \xrightarrow{e_3} X_3 \xrightarrow{e_2} X_2 \xrightarrow{e_1} X_1 \xrightarrow{e_0} X_0$$

has a (w.l.o.g. limit) cone of epimorphisms

Lemma A sufficient condition for an atomic topos $\text{Shut}(\mathbb{C})$ to satisfy DC is that every ω^{op} -diagram in \mathbb{C} has a cone.

Proposition The topos $\underline{\mathcal{P}}$ of probability sheaves satisfies DC.

Proof outline Consider any ω^{op} -diagram in SIBIP_0

$$\dots \xrightarrow{[p_4]} \Omega_4 \xrightarrow{[p_3]} \Omega_3 \xrightarrow{[p_2]} \Omega_2 \xrightarrow{[p_1]} \Omega_1 \xrightarrow{[p_0]} \Omega_0$$

Define $\Omega_\omega := \{(\omega_n)_{n \geq 0} \mid \forall i. p_i(\omega_{i+1}) = \omega_i\}$ (limit in SIBS)

By Daniell-Kolmogorov extension Ω_ω carries a unique probability measure that projects to each Ω_n , giving the required cone (limit in SIBIP). \square

iid sequences

Proposition For any $X \in \underline{RV}(A)$ there exists $S : (\underline{RV}(A))^{\mathbb{N}}$ s.t.
 $\forall n \in \mathbb{N}. S_n \sim X$ and S is a sequence of independent random variables.

Proof Define $S_0 := X$. Suppose we have S_0, \dots, S_{n-1} , where $n \geq 1$.

By the independence principle, there exists $S_n \in \underline{RV}(A)$

s.t. $S_n \sim X$ and $S_n \perp\!\!\!\perp (S_0, \dots, S_{n-1})$.

By DC, the S_n above can be found by a function $S : \mathbb{N} \rightarrow \underline{RV}(A)$. \square

By Countable product preservation $(\underline{RV}(A))^{\mathbb{N}} \cong \underline{RV}(A^{\mathbb{N}})$

So $S: \underline{RV}(A)^{\mathbb{N}}$ gives us $S' \cdot \underline{RV}(A^{\mathbb{N}})$ allowing us to express many properties, e.g.,

if $T \in \mathcal{B}_{A^{\mathbb{N}}}$ is tail then $\mathbb{P}_{S'}(T) = 0$ or $\mathbb{P}_{S'}(T) = 1$ (0-1 law)

We have a good setting for discrete-time stochastic processes.

Continuous time is more problematic. E.g.,

- What do we mean by $\underline{RV}(\mathbb{R}^{[0, \infty)})$?

We need to choose an SBS for $\mathbb{R}^{[0, \infty)}$ requiring pre-commitment to continuous processes (or similar) and a particular choice of σ -algebra, all of which encumbers the natural probabilistic development.

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Topos 2

(Randomness)

Random elements

Probability concerns random variables. The value of an A -valued random variable X varies according to the probability law P_X .

Randomness concerns random elements. A random element is a single fixed value of A obtained by sampling an A -valued random variable.

Random sequences

A random sequence is a random element in $2^{\mathbb{N}}$, obtained via an infinite sequence of fair coin tosses (i.e., by sampling the uniform probability distribution λ on $2^{\mathbb{N}}$).

$(01)^{\omega} := 0101010101010101\dots$ is not random
 $011010100010100010\dots$ is random!

$s \in 2^{\mathbb{N}}$ is naively random if, for every measurable $T \subseteq 2^{\mathbb{N}}$,

$$\lambda(T) = 1 \Rightarrow s \in T.$$

Problem No $s \in 2^{\mathbb{N}}$ is naively random.

Proof Take $T := 2^{\mathbb{N}} \setminus \{s\}$.

This problem is circumvented in approaches to randomness by, e.g., :

- restricting to T satisfying computability restrictions (algorithmic randomness)
- restricting to T definable in a given countable model (set theory)

Randomness-preserving functions

Even if inconsistent in itself, the notion of naive randomness suggests a sensible notion of randomness-preserving function.

A measurable function $f: \Omega' \rightarrow \Omega$ between standard Borel probability spaces is randomness preserving if

$$\forall T \in \mathcal{B}_\Omega. \mathbb{P}_\Omega(T) = 1 \Rightarrow \mathbb{P}_{\Omega'}(f^{-1}T) = 1$$

or \equiv by

$$\forall T \in \mathcal{B}_\Omega. \mathbb{P}_\Omega(T) = 0 \Rightarrow \mathbb{P}_{\Omega'}(f^{-1}T) = 0$$

Base categories

$\$IBR$ Standard Borel probability spaces + randomness-preserving functions

$\$IBR_0$ " " " " " " " mod 0

Covers

Let $f: \Omega' \rightarrow \Omega$ be randomness preserving

$T \in \mathcal{B}_\Omega$ is a subimage of f if, for all $S \in \mathcal{B}_\Omega$, $S \subseteq T$ and $\mathbb{P}_\Omega(S) > 0 \Rightarrow \mathbb{P}_{\Omega'}(f^{-1}S) > 0$.

An image of f is a subimage of maximum measure (amongst subimages)

A countable family $(f_n: \Omega_n \rightarrow \Omega)$ is covering if

$$\mathbb{P}_\Omega\left(\bigcup_n T_n\right) = 1 \quad \text{where each } T_n \text{ is an image of } f_n.$$

The random topos

Countable covering families form a Grothendieck topology both on $\mathcal{S}BIR$ and on $\mathcal{S}BIR_0$: the countable cover topology.

The random topos is defined by:

$$\underline{\mathcal{R}} := \text{Sh}_{cc}(\mathcal{S}BIR_0) \cong \text{Sh}_{cc}(\mathcal{S}BIR)$$

(sheaves for the countable cover topology)

- On $\mathbb{S}B\mathbb{R}_0$, countable cover = canonical = dense .
- \underline{R} is therefore a boolean topos (its internal logic is classical)
- DC holds
- The subobject classifier $\underline{\Omega}$

$$\underline{\Omega}(\Omega) := B_\Omega \text{ mod } \mathcal{O} \quad (\text{measure algebra})$$

- The real numbers \underline{R}

$$\underline{R}(\Omega) := \text{measurable functions } \Omega \rightarrow \mathbb{R} \text{ mod } \mathcal{O}$$

- Sequences $\underline{2^{\mathbb{N}}}$

$$\underline{2^{\mathbb{N}}}(\Omega) := \text{measurable functions } \Omega \rightarrow 2^{\mathbb{N}} \text{ mod } 0$$

- Random sequences $\underline{\text{Ran}} \subseteq \underline{2^{\mathbb{N}}}$

$$\underline{\text{Ran}}(\Omega) := \text{randomness-preserving functions } \Omega \rightarrow (2^{\mathbb{N}}, \lambda) \text{ mod } 0$$

$$\cong \text{SIBR}_0(\Omega, (2^{\mathbb{N}}, \lambda)) \quad \text{representable!}$$

Theorem Internally in $\underline{\mathbb{R}}$

$$\underline{\text{Ran}} = \left\{ s: 2^{\mathbb{N}} \mid \forall T \subseteq 2^{\mathbb{N}}, \lambda(T)=1 \text{ and } T \perp s \Rightarrow s \in T \right\}$$

a primitive relation of independence in $\underline{\mathbb{R}}$ related to Day convolution.

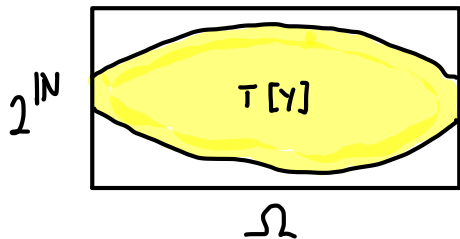
Theorem Internally in \mathbb{R} , there exists a translation-invariant probability measure $\underline{\lambda}: \mathcal{P}(\mathbb{Z}^{\mathbb{N}}) \rightarrow [0,1]$ extending the uniform Borel measure.

Outline construction of $\underline{\lambda}$

Suppose $T \in \mathcal{P}(\mathbb{Z}^{\mathbb{N}})(\Omega) \cong (\mathbb{Z}^{\mathbb{Z}^{\mathbb{N}}})(\Omega)$. Reindex T along $\Omega \otimes (\mathbb{Z}^{\mathbb{N}}, \lambda) \xrightarrow{\pi_1} \Omega$.

Consider $[Y] \in (\mathbb{Z}^{\mathbb{N}})(\Omega \otimes (\mathbb{Z}^{\mathbb{N}}, \lambda))$ where $Y := \Omega \otimes (\mathbb{Z}^{\mathbb{N}}, \lambda) \xrightarrow{\pi_2} \mathbb{Z}^{\mathbb{N}}$. (Generic random sequence)

Then $T[Y] \in \mathbb{Z}(\Omega \otimes (\mathbb{Z}^{\mathbb{N}}, \lambda)) \cong \mathcal{B}_{\Omega \times \mathbb{Z}^{\mathbb{N}}} \text{ mod } \mathcal{O}$



$$\begin{aligned} \underline{\lambda}(T) &:= [\omega \mapsto \lambda \{s \in \mathbb{Z}^{\mathbb{N}} \mid (\omega, s) \in T\}] \\ &: \Omega \rightarrow [0,1] \text{ mod } \mathcal{O} = \underline{[0,1]}(\Omega) \end{aligned}$$

More generally all Borel probability measures extend to (canonical) powerset measures. We state the precise theorem without explaining the underlined concepts, which rely on a general theory of randomness in \mathbb{R} .

Theorem Internally in \mathbb{R} , for every standard Borel probability space $(A, \mathcal{B}, \mu) : \mathcal{B} \rightarrow [0, 1]$, there exists a unique probability measure $\mu^* : \mathcal{P}(A) \rightarrow [0, 1]$ satisfying:

- μ^* extends μ ,
- μ^* is near Borel,
- there are enough μ^* -random elements, and
- μ^* -random elements are Borel testable.

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Topos 3

(Probability again)

The topos of random probability sheaves

The topos

$\underline{\mathcal{P}}_{\underline{R}}$:= the topos of probability sheaves $\underline{\mathcal{P}}$
relative to the random topos \underline{R}

One can externalise this to a Grothendieck topos over $\underline{\text{Set}}$,
but we shall study $\underline{\mathcal{P}}_{\underline{R}}$ from the internal perspective of \underline{R} .

The RV endofunctor

$$\underline{RV} : \underline{P}_{\underline{R}} \rightarrow \underline{P}_{\underline{R}}$$

$$\begin{array}{ccc}
 \Omega & & \underline{RV}(\underline{A})(\Omega) := \text{arbitrary functions } \Omega \rightarrow \underline{A}(\Omega) \text{ mod } 0 \\
 \uparrow [p] & \mapsto & \downarrow [f] \mapsto [\omega' \mapsto f(p(\omega')) \cdot [p]] \\
 \Omega' & & \underline{RV}(\underline{A})(\Omega')
 \end{array}$$

$f \equiv f' \text{ mod } 0 \iff \mathbb{P}_{\Omega}^* (\{\omega \mid f(\omega) \neq f'(\omega)\}) = 0$
 is well-defined because $\mathbb{P}_{\Omega}^* : \mathcal{G}(\Omega) \rightarrow [0, 1]$

The law of a random variable

The distributions endofunctor $\underline{D} : \underline{P}_{\mathbb{R}} \rightarrow \underline{P}_{\mathbb{R}}$ (in fact monad)

$$\underline{D}(\underline{A}) := \{ \mu : \mathcal{P}(\underline{A}) \rightarrow [0,1] \mid \mu \text{ a probability measure} \}$$

(powerset probability measures, internally defined in $\underline{P}_{\mathbb{R}}$)

Probability law natural transformation $\mathbb{P} : \underline{RV} \Rightarrow \underline{D}$

$$\mathbb{P}_{\underline{A}} : \underline{RV}(\underline{A})(\Omega) \rightarrow \underline{D}(\underline{A})(\Omega)$$

$$[x : \Omega \rightarrow \underline{A}(\Omega)] \mapsto (B \in \mathcal{P}(\underline{A})(\Omega)) \mapsto \mathbb{P}_{\Omega}^*(x^{-1}B_{\Omega})$$

Properties of the set-up in $\underline{P}_{\mathbb{R}}$

- \underline{RV} is faithful and preserves countable limits
- \underline{D} is faithful and taut
- $\mathbb{P}, \underline{RV} \Rightarrow \underline{D}$ is taut
- The independence principle & invariance principle
- DC

This supports the development of an axiomatic synthetic probability theory based on the \underline{RV} functor.

Random variables can be valued in arbitrary sets and their probability laws are powerset measures.

We now have a very natural route to continuous-time stochastic processes; e.g.,

$\underline{RV}(\mathbb{R})^{[0, \infty)}$ — processes up to modification equivalence

$\underline{RV}(\mathbb{R}^{[0, \infty)})$ — processes up to indistinguishability

Given $X: \underline{RV}(\mathbb{R}^{[0, \infty)})$,

$\exists Y: \underline{RV}(\mathbb{R}^{[0, \infty)}) . \mathbb{P}_Y(C[0, \infty)) = 1 \wedge \forall t \in [0, \infty) X_t = Y_t$

Says that X has a continuous modification.

A fourth topos!

- A nominal approach to probabilistic separation logic,
Li, Ahmed, Aytac, Holtzen, Johnson-Freyd, LICS 2024.

Considers the atomic topos on the subcategory of \mathcal{SIBIR}_0 consisting of maps that are singleton covers. This is proven equivalent to a category of continuous group actions, and several probabilistically-interesting constructions are related across this equivalence.

(cf. Schanuel topos \equiv nominal sets)

Paper + on-line talks

- Equivalence and conditional independence in atomic sheaf logic, LICS 2024.
Study of the independence and invariance principles in atomic sheaf categories.
- Probability sheaves, Topos à l'IHES, 2015.
An early talk on probability sheaves.
- A mathematical theory of true randomness, Parts 1 & 2. UNISA seminar, 2023.
An axiomatic theory of randomness & measure, modelled by the random topos.
- Synthetic probability theory. Talk at Categorical Probability & Statistics, 2020
An axiomatic development of the synthetic probability theory modelled by $\underline{\mathbb{P}}_{\mathbb{R}}$.