Polynomial Interfaces

Topos Colloquium

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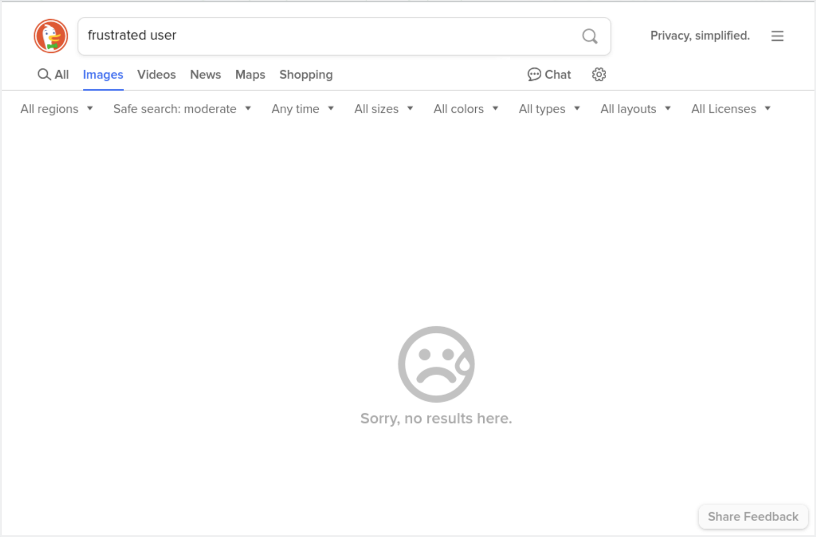
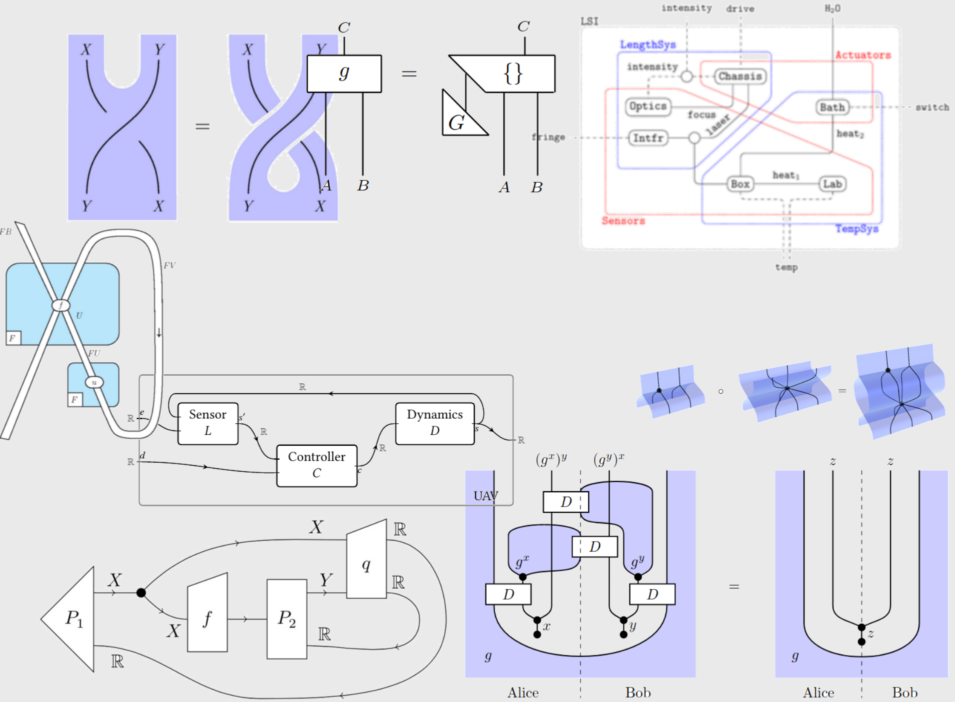
U.S. DEPARTMENT OF COMMERCE

User interface





X
Y
Y
X
=
C
g
A
B
=
C
{}
G
LSI
intensity
drive
H2O
LengthSys
Actuators
intensity
Chassis
Optics
focus
laser
Bath
switch
fringe
Intfr
heat2
Box
heat1
Lab
Sensors
TempSys
temp
FB
FV
U
F
FU
u
F
ℝ
Sensor
L
Dynamics
D
Controller
C
UAV
(g^x)^y
(g^y)^x
D
g^x
g^y
x
y
g
Alice
Bob
=
z
z
g
Alice
Bob
X
ℝ
P1
f
P2
Y
q

```haskell
tryCapitalizeM :: (Functor m, Monad m, Walkable Inline a, Default a, Eq a) =>
        (String -> m a) -> String -> Bool -> m a
tryCapitalizeM f varname capitalize
  | capitalize = do
    res <- f (capitalizeFirst varname)
    case res of
      xs | xs == def -> f varname >>= walkM capStrFst
         | otherwise -> return xs
  | otherwise  = f varname
  where
    capStrFst (Str s) = return $ Str $ capitalizeFirst s
    capStrFst x = retu
```

```
schema Spec = literal : Type {
    imports SpecEnums
    entities
    // User-defined entities
        Spec Profile Limit
    // Computed entities
        Filter
    foreign_keys
        process      : Spec -> Process
        removalReq   : Spec -> RemovalReq
        lay          : Spec -> Lay

        filterType       : Filter -> FilterType
        shortWaveFilter  : Filter -> ShortWaveFilter
        samplingLength   : Filter -> SamplingLength

                     : Profile -> Spec
                     : Profile -> Rule
                     : Profile -> ParamSymbol
                     : Profile -> Filter

                     : Limit -> Profile
                     : Limit -> Orientation

   lReq = Spec.process.processReq

                         : Spec -> String
                         : Spec -> String
                         : Spec -> String
                         : Spec -> String
```

```julia
# Commutative squares
#####################

A, B, C, D, X, Y = Ob(FreeCategory, :A, :B, :C, :D, :X, :Y)
f, g, m, n = Hom(:f, A, C), Hom(:g, B, D), Hom(:m, A, B), Hom(:n, C, D)

α = SquareDiagram(m, n, f, g)

h, k, p = Hom(:h, C, X),Hom(:g, D, Y), Hom(:p, X, Y)
β = SquareDiagram(n, p, h, k)
@test compose(α, β) == SquareDiagram(m, p, f⋅h, g⋅k)

@test_throws ErrorException pcompose(α, α)
@test pcompose(pid(src(α)), α) == α
@test pcompose(α, pid(tgt(α))) == α
```

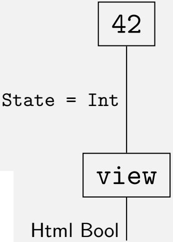
Some theses:

- Applied category theory needs better UIs.
- Categorical abstractions can be useful for UI.
- UI is an accessible entrypoint for categorical ideas.
- UI is a toy model for categorical systems theory.

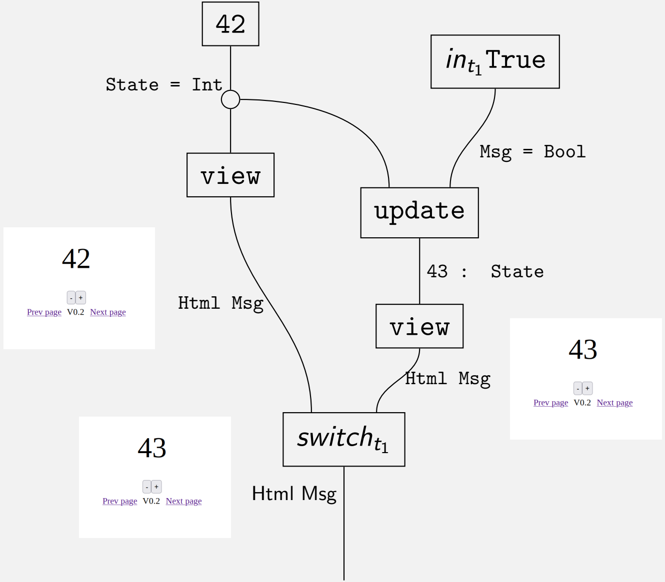
Elm

Demo

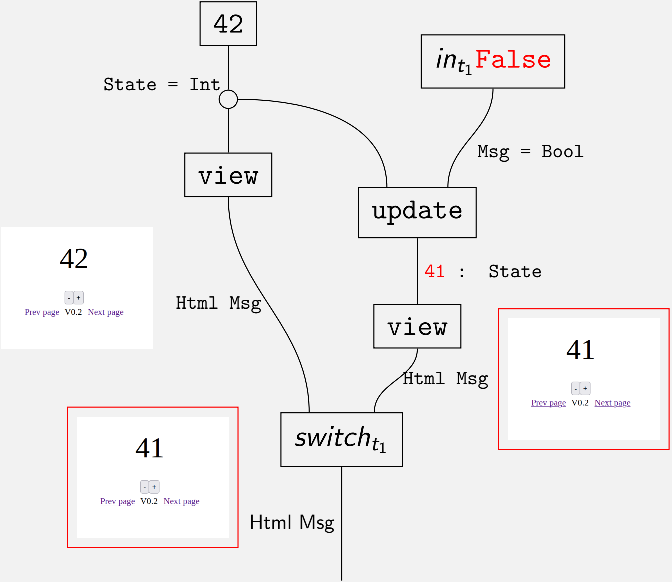
Counter, $t = 0$



Counter, t = 1

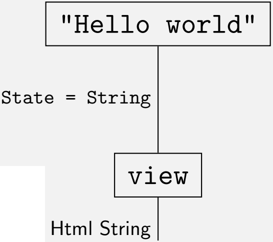

42
in_{t_1} True
State = Int
Msg = Bool
view
update
43 : State
view
42
- +
Prev page V0.2 Next page
Html Msg
Html Msg
43
- +
Prev page V0.2 Next page
$switch_{t_1}$
43
- +
Prev page V0.2 Next page
Html Msg

Counter, $t = 1'$



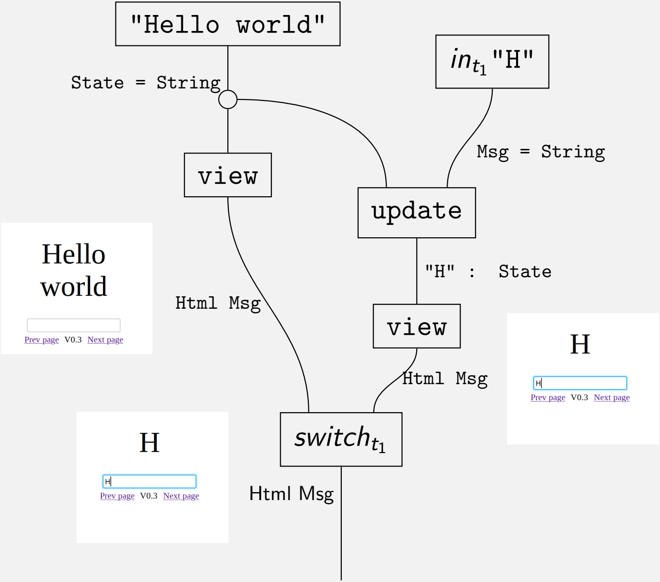
42
State = Int
in_{t_1} False
Msg = Bool
view
update
41 : State
Html Msg
view
Html Msg
$switch_{t_1}$
Html Msg
42
- +
Prev page V0.2 Next page
41
- +
Prev page V0.2 Next page
41
- +
Prev page V0.2 Next page

Input, $t = 0$

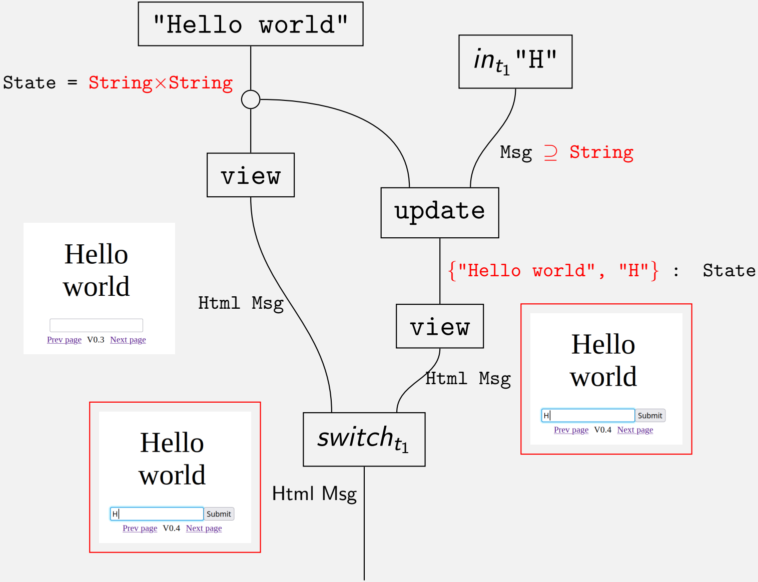

"Hello world"
State = String
view
Html String


Hello world
Prev page V0.3 Next page

Input, t = 1

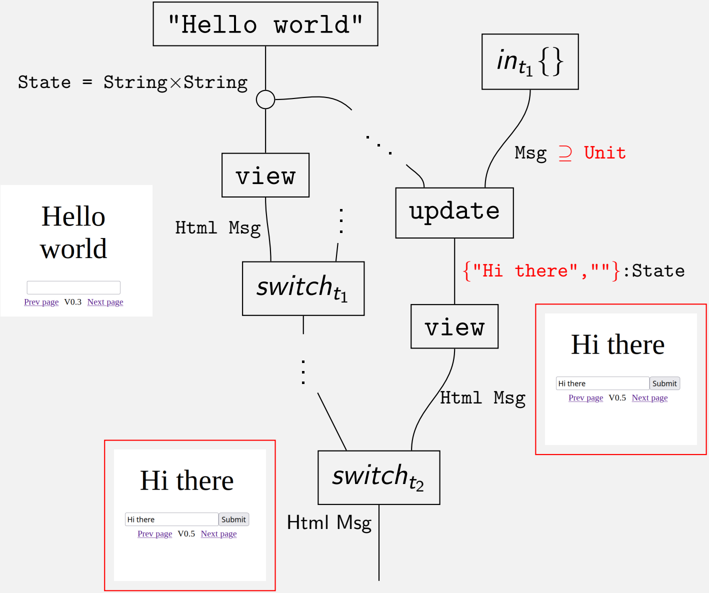

"Hello world"
State = String
in_{t_1} "H"
Msg = String
view
update
"H" : State
Hello world
Prev page V0.3 Next page
Html Msg
view
Html Msg
H
H
Prev page V0.3 Next page
$switch_{t_1}$
H
H
Prev page V0.3 Next page
Html Msg

Input′, t = 1

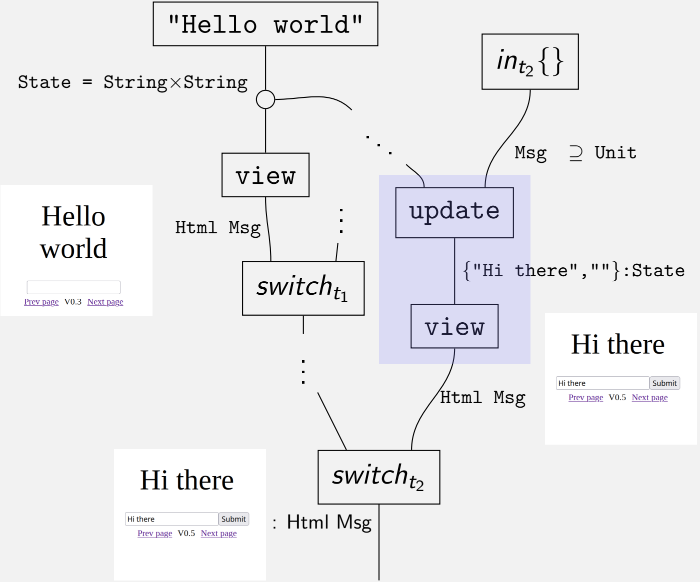


"Hello world"
State = String×String
in_{t_1} "H"
view
Msg ⊇ String
update
{"Hello world", "H"} : State
Hello world
Prev page V0.3 Next page
Html Msg
view
Html Msg
Hello world
H
Submit
Prev page V0.4 Next page
$switch_{t_1}$
Html Msg
Hello world
H
Submit
Prev page V0.4 Next page

Input′, t = 2


"Hello world"
State = String×String
$in_{t_1}\{\}$
Msg ⊇ Unit
view
update
Html Msg
$switch_{t_1}$
{"Hi there",""}:State
view
Hello world
Prev page V0.3 Next page
Hi there
Hi there Submit
Prev page V0.5 Next page
Html Msg
$switch_{t_2}$
Hi there
Hi there Submit
Prev page V0.5 Next page
Html Msg

The lens pattern



Interface loops

An *interface loop* is a natural transformation from a category to a polynomial.

$$\mathbb{C} \xrightarrow{\quad \ell \quad} \mathbf{P}$$

Types ≈ Numbers

$0 =$ "impossibility"

$1 =$ "necessity"

$+ =$ "or"

$\times =$ "and"

$\underline{m} + \underline{n} = \underline{m + n}$

$\underline{m} \times \underline{n} = \underline{m \times n}$

$$\underline{n} = \underbrace{1 + \cdots + 1}_{n \text{ times}}$$

Types ≈ Numbers

$0 =$ “impossibility”

$1 =$ “necessity”

$+ =$ “or”

$\times =$ “and”

$\texttt{Msg = Str + 1}$

$\texttt{State = Str} \times \texttt{Str}$

$$\texttt{Str} \;=\; \underbrace{1 + \cdots + 1}_{\text{String}-\text{many times}} \;=\; 1 + \texttt{Char} \times \texttt{Str}$$

Types ≈ Numbers

$0 =$ "impossibility"

$1 =$ "necessity"

$+ =$ "or"

$\times =$ "and"

`Msg = Str + 1`

$\texttt{State = Str} \times \texttt{Str}$

$$\texttt{Str} \; = \; \underbrace{1 + \cdots + 1}_{\text{String−many times}} \; = \; 1 + \texttt{Char} \times \texttt{Str}$$

$$\frac{\texttt{A + B} \to \texttt{C}}{\texttt{A} \to \texttt{C} \qquad \texttt{B} \to \texttt{C}}$$

$$\frac{\texttt{C} \to \texttt{A} \times \texttt{B}}{\texttt{C} \to \texttt{A} \qquad \texttt{C} \to \texttt{B}}$$

Types ≈ Numbers

$0 =$ "impossibility"

$1 =$ "necessity"

$+ =$ "or"

$\times =$ "and"

$\texttt{Msg = Str + 1}$

$\texttt{State = Str} \times \texttt{Str}$

$$\texttt{Str} \;=\; \underbrace{1 + \cdots + 1}_{\text{String}-\text{many times}} \;=\; 1 + \texttt{Char} \times \texttt{Str}$$

$$\frac{\sum_{\texttt{i}} \texttt{A}_{\texttt{i}} \to \texttt{C}}{\forall \texttt{i},\;\; \texttt{A}_{\texttt{i}} \to \texttt{C}}$$

$$\frac{\texttt{C} \to \prod_{\texttt{i}} \texttt{A}_{\texttt{i}}}{\forall \texttt{i},\;\; \texttt{C} \to \texttt{A}_{\texttt{i}}}$$

Types ≈ Numbers

$0 =$ "impossibility"

$1 =$ "necessity"

$+ =$ "or"

$\times =$ "and"

$\texttt{Msg = Str + 1}$

$\texttt{State = Str} \times \texttt{Str}$

$$\texttt{Str} \;=\; \underbrace{1 + \cdots + 1}_{\text{String}-\text{many times}} \;=\; 1 + \texttt{Char} \times \texttt{Str}$$

$\sum_{\texttt{i}} \texttt{A} \cong \texttt{I} \times \texttt{A}$

$\prod_{\texttt{i}} \texttt{A} \cong \texttt{A}^{\texttt{I}}$

Types $\approx$ Numbers

$$\texttt{Str} = 1 + \texttt{Char} \times \texttt{Str}$$

Types ≈ Numbers

$$\begin{aligned} \texttt{Str} &= 1 + \texttt{Char} \times \texttt{Str} \\ &= 1 + \texttt{Char} \times (1 + \texttt{Char} \times \texttt{Str}) \end{aligned}$$

Types ≈ Numbers

$$\begin{aligned} \texttt{Str} &= 1 + \texttt{Char} \times \texttt{Str} \\ &= 1 + \texttt{Char} \times (1 + \texttt{Char} \times \texttt{Str}) \\ &= 1 + (\texttt{Char} \times 1) + (\texttt{Char} \times \texttt{Char} \times \texttt{Str}) \end{aligned}$$

Types $\approx$ Numbers

$$
\begin{aligned}
\texttt{Str} &= 1 + \texttt{Char} \times \texttt{Str} \\
&= 1 + \texttt{Char} \times (1 + \texttt{Char} \times \texttt{Str}) \\
&= 1 + (\texttt{Char} \times 1) + (\texttt{Char} \times \texttt{Char} \times \texttt{Str}) \\
&= 1 + \texttt{Char} + \texttt{Char}^2 \times (1 + \texttt{Char} \times \texttt{Str})
\end{aligned}
$$

Types ≈ Numbers

$$
\begin{aligned}
\texttt{Str} &= 1 + \texttt{Char} \times \texttt{Str} \\
&= 1 + \texttt{Char} \times (1 + \texttt{Char} \times \texttt{Str}) \\
&= 1 + (\texttt{Char} \times 1) + (\texttt{Char} \times \texttt{Char} \times \texttt{Str}) \\
&= 1 + \texttt{Char} + \texttt{Char}^2 \times (1 + \texttt{Char} \times \texttt{Str}) \\
&\vdots \\
&= 1 + \texttt{Char} + \texttt{Char}^2 + \texttt{Char}^3 + \ldots
\end{aligned}
$$

Types ≈ Numbers

$$\begin{aligned}
\mathtt{Str} &= 1 + \mathtt{Char} \times \mathtt{Str} \\
&= 1 + \mathtt{Char} \times (1 + \mathtt{Char} \times \mathtt{Str}) \\
&= 1 + (\mathtt{Char} \times 1) + (\mathtt{Char} \times \mathtt{Char} \times \mathtt{Str}) \\
&= 1 + \mathtt{Char} + \mathtt{Char}^2 \times (1 + \mathtt{Char} \times \mathtt{Str}) \\
&\vdots \\
&= 1 + \mathtt{Char} + \mathtt{Char}^2 + \mathtt{Char}^3 + \dots
\end{aligned}$$

A string is empty <u>or</u> a character c_1

<u>or</u> a pair of characters c_1 <u>and</u> c_2

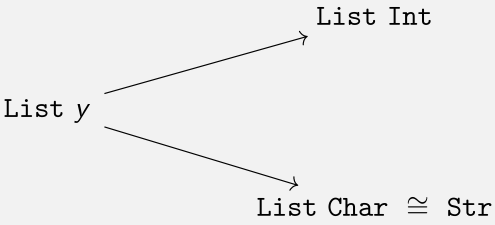
<u>or</u> ...

Numbers:Types::Functions:Functors

A list of y's is empty or a singleton y_1

or a pair y_1 and y_2

$\vdots$

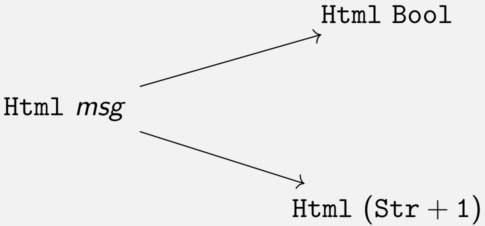


Numbers:Types::Functions:Functors

An HTML *msg* block is a list of display properties

<u>and</u> a list of *msg* callbacks

<u>and</u> a list of HTML *msg* blocks



Numbers:Types::Functions:Functors

Constants $\quad 2 : \texttt{Int} \to \texttt{Int}$ $\qquad$ $\texttt{Int} : \textbf{Type} \to \textbf{Type}$

$n \mapsto 2$ $\qquad$ $A \mapsto \texttt{Int}$

Numbers:Types::Functions:Functors

Constants	$2 : \texttt{Int} \to \texttt{Int}$	$\texttt{Int} : \textbf{Type} \to \textbf{Type}$
	$n \mapsto 2$	$A \mapsto \texttt{Int}$
Identities	$x : \texttt{Int} \to \texttt{Int}$	$y : \textbf{Type} \to \textbf{Type}$
	$n \mapsto n$	$A \mapsto A$

Numbers:Types::Functions:Functors

Constants	$2 : \texttt{Int} \to \texttt{Int}$ $n \mapsto 2$	$\texttt{Int} : \textbf{Type} \to \textbf{Type}$ $A \mapsto \texttt{Int}$
Identities	$x : \texttt{Int} \to \texttt{Int}$ $n \mapsto n$	$y : \textbf{Type} \to \textbf{Type}$ $A \mapsto A$
Linear	$2x : \texttt{Int} \to \texttt{Int}$ $n \mapsto 2n$	$2y : \textbf{Type} \to \textbf{Type}$ $A \mapsto A + A$

Numbers:Types::Functions:Functors

Constants	$2 : \texttt{Int} \to \texttt{Int}$ $n \mapsto 2$	$\texttt{Int} : \textbf{Type} \to \textbf{Type}$ $A \mapsto \texttt{Int}$
Identities	$x : \texttt{Int} \to \texttt{Int}$ $n \mapsto n$	$y : \textbf{Type} \to \textbf{Type}$ $A \mapsto A$
Linear	$2x : \texttt{Int} \to \texttt{Int}$ $n \mapsto 2n$	$2y : \textbf{Type} \to \textbf{Type}$ $A \mapsto A + A$
Power	$x^2 : \texttt{Int} \to \texttt{Int}$ $n \mapsto n^2$	$y^2 : \textbf{Type} \to \textbf{Type}$ $A \mapsto A \times A$

Numbers:Types::Functions:Functors

Constants	$2 : \texttt{Int} \to \texttt{Int}$ $n \mapsto 2$	$\texttt{Int} : \textbf{Type} \to \textbf{Type}$ $A \mapsto \texttt{Int}$
Identities	$x : \texttt{Int} \to \texttt{Int}$ $n \mapsto n$	$y : \textbf{Type} \to \textbf{Type}$ $A \mapsto A$
Linear	$kx : \texttt{Int} \to \texttt{Int}$ $n \mapsto kn$	$By : \textbf{Type} \to \textbf{Type}$ $A \mapsto \sum_b A \cong B \times A$
Power	$x^k : \texttt{Int} \to \texttt{Int}$ $n \mapsto n^k$	$y^B : \textbf{Type} \to \textbf{Type}$ $A \mapsto \prod_b A \cong A^B$

3 perspectives on polynomials

	Graphical	Algebraic	Combinatorial
Constants	$\overbrace{\bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet}^{A\text{-many copies}}$	A	$\emptyset$ $\downarrow$ A

3 perspectives on polynomials

	Graphical	Algebraic	Combinatorial
Constants	$\overbrace{\bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet}^{A\text{-many copies}}$	A	$\begin{matrix}\emptyset \\ \downarrow \\ A\end{matrix}$
Variables	$\begin{matrix}\uparrow \\ \bullet\end{matrix}$	y	$\begin{matrix}1 \\ \downarrow \\ 1\end{matrix}$

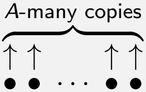
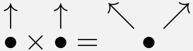
3 perspectives on polynomials

	Graphical	Algebraic	Combinatorial
Constants	$\overbrace{\bullet\ \ \bullet\ \ \cdots\ \ \bullet\ \ \bullet}^{A\text{-many copies}}$	A	$\begin{array}{c}\emptyset\\ \downarrow\\ A\end{array}$
Variables	$\overset{\uparrow}{\bullet}$	y	$\begin{array}{c}1\\ \downarrow\\ 1\end{array}$
Linear	$\overset{\uparrow}{\bullet} + \overset{\uparrow}{\bullet} = \overset{\uparrow}{\bullet}\ \overset{\uparrow}{\bullet}$	$2y$	$\begin{array}{c}2\\ \downarrow\\ 2\end{array}$

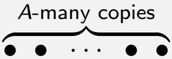
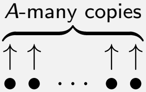
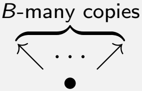
3 perspectives on polynomials

	Graphical	Algebraic	Combinatorial
Constants	$\overbrace{\bullet\ \ \bullet\ \ \cdots\ \ \bullet\ \ \bullet}^{A\text{-many copies}}$	A	$\emptyset$ $\downarrow$ A
Variables	$\uparrow$ $\bullet$	y	1 $\downarrow$ 1
Linear	$\overbrace{\overset{\uparrow}{\bullet}\ \overset{\uparrow}{\bullet}\ \cdots\ \overset{\uparrow}{\bullet}\ \overset{\uparrow}{\bullet}}^{A\text{-many copies}}$	Ay	A $\downarrow$ A

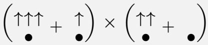
3 perspectives on polynomials

	Graphical	Algebraic	Combinatorial
Constants		A	$\begin{array}{c}\emptyset\\ \downarrow\\ A\end{array}$
Variables		y	$\begin{array}{c}1\\ \downarrow\\ 1\end{array}$
Linear		Ay	$\begin{array}{c}A\\ \downarrow\\ A\end{array}$
Power		y^2	$\begin{array}{c}2\\ \downarrow\\ 1\end{array}$

3 perspectives on polynomials

	Graphical	Algebraic	Combinatorial
Constants		A	$\emptyset \downarrow A$
Variables		y	$1 \downarrow 1$
Linear		Ay	$A \downarrow A$
Power		y^B	$B \downarrow 1$

3 perspectives on polynomials

Graphical	Algebraic	Combinatorial
	$(y^3 + y) \times (y^2 + 1)$	$\begin{pmatrix} 3 & & 1 \\ \downarrow & + & \downarrow \\ 1 & & 1 \end{pmatrix} \times \begin{pmatrix} 2 & & 0 \\ \downarrow & + & \downarrow \\ 1 & & 1 \end{pmatrix}$

3 perspectives on polynomials

Graphical	Algebraic	Combinatorial
$\left(\overset{\uparrow\uparrow\uparrow}{\bullet} + \overset{\uparrow}{\bullet}\right) \times \left(\overset{\uparrow\uparrow}{\bullet} + \bullet\right)$	$(y^3 + y) \times (y^2 + 1)$	$\begin{pmatrix}3 & & 1\\ \downarrow & + & \downarrow\\ 1 & & 1\end{pmatrix} \times \begin{pmatrix}2 & & 0\\ \downarrow & + & \downarrow\\ 1 & & 1\end{pmatrix}$
$= \left(\overset{\uparrow\uparrow\uparrow}{\bullet} \times \overset{\uparrow\uparrow}{\bullet}\right) + \left(\overset{\uparrow\uparrow\uparrow}{\bullet} \times \bullet\right)$	$= (y^3 \times y^2) + (y^3 \times 1)$	$= \begin{pmatrix}3 & & 2\\ \downarrow & \times & \downarrow\\ 1 & & 1\end{pmatrix} + \begin{pmatrix}3 & & 0\\ \downarrow & \times & \downarrow\\ 1 & & 1\end{pmatrix}$
$+ \left(\overset{\uparrow}{\bullet} \times \overset{\uparrow\uparrow}{\bullet}\right) + \left(\overset{\uparrow}{\bullet} \times \bullet\right)$	$+ (y \times y^2) + (y \times 1)$	$+ \begin{pmatrix}1 & & 2\\ \downarrow & \times & \downarrow\\ 1 & & 1\end{pmatrix} + \begin{pmatrix}1 & & 0\\ \downarrow & \times & \downarrow\\ 1 & & 1\end{pmatrix}$

3 perspectives on polynomials

Graphical	Algebraic	Combinatorial
$\left(\overset{\uparrow\uparrow\uparrow}{\bullet} + \overset{\uparrow}{\bullet}\right) \times \left(\overset{\uparrow\uparrow}{\bullet} + \bullet\right)$	$(y^3 + y) \times (y^2 + 1)$	$\left(\underset{1}{\overset{3}{\downarrow}} + \underset{1}{\overset{1}{\downarrow}}\right) \times \left(\underset{1}{\overset{2}{\downarrow}} + \underset{1}{\overset{0}{\downarrow}}\right)$
$= \left(\overset{\uparrow\uparrow\uparrow}{\bullet} \times \overset{\uparrow\uparrow}{\bullet}\right) + \left(\overset{\uparrow\uparrow\uparrow}{\bullet} \times \bullet\right)$	$= (y^3 \times y^2) + (y^3 \times 1)$	$= \left(\underset{1}{\overset{3}{\downarrow}} \times \underset{1}{\overset{2}{\downarrow}}\right) + \left(\underset{1}{\overset{3}{\downarrow}} \times \underset{1}{\overset{0}{\downarrow}}\right)$
$+ \left(\overset{\uparrow}{\bullet} \times \overset{\uparrow\uparrow}{\bullet}\right) + \left(\overset{\uparrow}{\bullet} \times \bullet\right)$	$+ (y \times y^2) + (y \times 1)$	$+ \left(\underset{1}{\overset{1}{\downarrow}} \times \underset{1}{\overset{2}{\downarrow}}\right) + \left(\underset{1}{\overset{1}{\downarrow}} \times \underset{1}{\overset{0}{\downarrow}}\right)$
$= \overset{\uparrow\uparrow\uparrow\uparrow\uparrow}{\bullet} + \overset{\uparrow\uparrow\uparrow}{\bullet} + \overset{\uparrow\uparrow\uparrow}{\bullet} + \overset{\uparrow}{\bullet}$	$= y^5 + 2y^3 + y$	$= \underset{1}{\overset{5}{\downarrow}} + \underset{2}{\overset{3+3}{\downarrow}} + \underset{1}{\overset{0}{\downarrow}} = \underset{4}{\overset{11}{\downarrow}}$

Polynomial functors

A **polynomial functor** is a sum of products of y.

Non-example: 2^y

$$\mathbf{P} = A_0 + A_1 y + A_2 y^2 + \ldots + \overbrace{A_B y^B}^{\text{monomial}}$$

Notation and terminology:

"Positions": $\mathbf{P}_0 := A_0 + A_1 + \ldots + A_B = \mathbf{P}(y=1)$

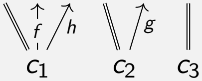
"Directions": $\mathbf{P}'_a := B$ for $a \in A_B \subseteq \mathbf{P}_0$

$\mathbf{P}' := \sum_{a \in \mathbf{P}_0} \mathbf{P}'_a$

$$\text{i.e.,} \quad \mathbf{P} \;=\; \begin{array}{c}\mathbf{P}' \\ \downarrow \\ \mathbf{P}_0\end{array} \;\approx\; \{\mathbf{P}'_a\}_{a \in \mathbf{P}_0}$$

Categories as polynomials

$$\mathbb{C} = \left\{ \begin{tikzcd} c_1 \arrow[r, "f"'] \arrow[rr, bend left, "h=g\circ f"] & c_2 \arrow[r, "g"'] & c_3 \end{tikzcd} \right\}$$

Graphical	Algebraic	Combinatorial
	$\sum_{c \in \mathrm{Ob}(\mathbb{C})} y^{\mathrm{Hom}(c,-)}$	$\mathrm{Ar}(\mathbb{C}) \xrightarrow{\mathrm{dom}} \mathrm{Ob}(\mathbb{C})$

What is a polynomial?

A position is a...	**and a direction is a...**
Question	Answer
Problem	Solution
Point (in space)	Tangent
Input	Output
State (of a game)	Available play

From Libkind & Spivak

Polynomial transforms

$$\mathbf{P} \xrightarrow{\quad\mathbf{f}\quad} \mathbf{Q}$$

$$\begin{array}{ccccc}
\mathbf{P}' & \xleftarrow{\;\mathbf{f}'\;} & \mathbf{Q}'_{\mathbf{f}} & \longrightarrow & \mathbf{Q}' \\
\downarrow & & \downarrow & \lrcorner & \downarrow \\
\mathbf{P}_0 & \longrightarrow & \mathbf{P}_0 & \xrightarrow[\;\mathbf{f}_0\;]{} & \mathbf{Q}_0
\end{array}$$

Polynomial transforms

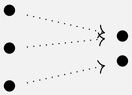
$$\mathbf{P} \xrightarrow{\quad\quad\quad\quad\quad\quad \mathbf{f} \quad\quad\quad\quad\quad\quad} \mathbf{Q}$$

$$\begin{array}{ccccc}
\mathbf{P}'_a & \xleftarrow{\quad \mathbf{f}'_a \quad} & \mathbf{Q}'_{\mathbf{f}_0(a)} & & \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \\
\mathbf{P}' & \xleftarrow{\quad \mathbf{f}' \quad} & \mathbf{Q}'_{\mathbf{f}} & \longrightarrow \lrcorner & \mathbf{Q}' \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{P}_0 & \longrightarrow & \mathbf{P}_0 & \xrightarrow[\mathbf{f}_0]{} & \mathbf{Q}_0
\end{array}$$

Transforms: constants

$\mathbf{P} = A$ $\mathbf{Q} = B$

Graphical



Algebraic

$$\begin{array}{ccc} A & \xrightarrow{\mathbf{f}_0} & B \\ \| & & \| \\ A & \xrightarrow[\mathbf{f}_0]{} & B \end{array}$$

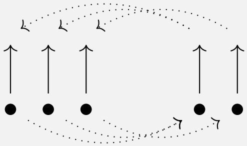
Combinatorial

$$\begin{array}{ccccc} \emptyset & \xleftarrow{!} & \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ A & = & A & \xrightarrow[\mathbf{f}_0]{} & B \end{array}$$

Transforms: linear

$\mathbf{P} = Ay$ $\qquad\qquad$ $\mathbf{Q} = By$

Graphical



Algebraic

$$\begin{array}{ccc} A \times X & \xrightarrow{\mathbf{f}_0 \times X} & B \times X \\ {\scriptstyle A\times k}\downarrow & & \downarrow{\scriptstyle B\times k} \\ A \times Y & \xrightarrow[\mathbf{f}_0 \times Y]{} & B \times Y \end{array}$$

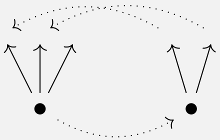
Combinatorial

$$\begin{array}{ccccc} A & = & A & \xrightarrow{\mathbf{f}_0} & B \\ \| & & \| \;\lrcorner & & \| \\ A & = & A & \xrightarrow[\mathbf{f}_0]{} & B \end{array}$$

Transforms: power

$\mathbf{P} = y^A$ $\mathbf{Q} = y^B$

Graphical



Algebraic

$$\begin{array}{ccc} X^A & \xrightarrow{-\circ \mathbf{f}'} & X^B \\ {\scriptstyle k\circ -}\downarrow & & \downarrow{\scriptstyle k\circ -} \\ Y^A & \xrightarrow[-\circ \mathbf{f}']{} & Y^B \end{array}$$

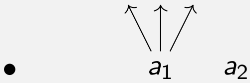
Combinatorial

$$\begin{array}{ccccc} A & \xleftarrow{\mathbf{f}'} & B & = & B \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ 1 & = & 1 & = & 1 \end{array}$$

Transforms: terminal states

$\text{Hom}(1, \mathbf{P}) \cong \mathbf{P}(0) = A_0$

Graphical



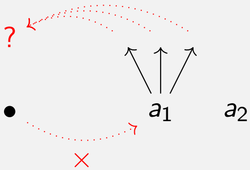
Combinatorial

$$\begin{array}{ccccc} \emptyset & \xleftarrow{\mathbf{f}'} & \bullet & \longrightarrow & 3 \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ 1 & = & 1 & \xrightarrow[\mathbf{f}_0]{} & 2 \end{array}$$

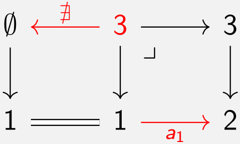
Transforms: terminal states

$\mathrm{Hom}(1, \mathbf{P}) \cong \mathbf{P}(0) = A_0$

Graphical



Combinatorial



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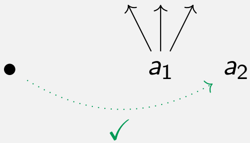
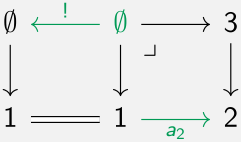
Graphical



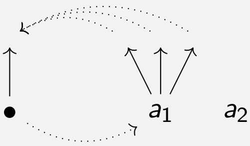
Combinatorial



Transforms: positions

$\mathrm{Hom}(y, \mathbf{P}) \cong \mathbf{P}_0$

Graphical



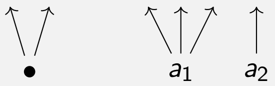
Combinatorial

$$
\begin{array}{ccccc}
1 & \xleftarrow{!} & \mathbf{P}'_a & \longrightarrow & \mathbf{P}' \\
\downarrow & & \downarrow & \lrcorner & \downarrow \\
1 & = & 1 & \xrightarrow[a]{} & \mathbf{P}_0
\end{array}
$$

Transforms: Yoneda

$\mathrm{Hom}(y^B, \mathbf{P}) \cong \mathbf{P}(B)$

Graphical



Combinatorial

$$
\begin{array}{ccccc}
B & \xleftarrow{\mathbf{f}'} & \mathbf{P}'_a & \longrightarrow & \mathbf{P}' \\
\downarrow & & \downarrow & \lrcorner & \downarrow \\
1 & = & 1 & \xrightarrow[a]{} & \mathbf{P}_0
\end{array}
$$

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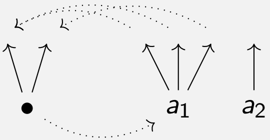
$\mathrm{Hom}(y^B, \mathbf{P}) \cong \mathbf{P}(B)$

Graphical



Combinatorial

$$\begin{array}{ccccc} B & \xleftarrow{\mathbf{f}'} & \mathbf{P}'_a & \longrightarrow & \mathbf{P}' \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ 1 & = & 1 & \xrightarrow[a]{} & \mathbf{P}_0 \end{array}$$

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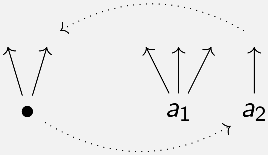
$\mathrm{Hom}(y^B, \mathbf{P}) \cong \mathbf{P}(B)$

Graphical



Combinatorial

$$\begin{array}{ccccc}
B & \xleftarrow{\mathbf{f}'} & \mathbf{P}'_a & \longrightarrow & \mathbf{P}' \\
\downarrow & & \downarrow & \lrcorner & \downarrow \\
1 & = & 1 & \xrightarrow[a]{} & \mathbf{P}_0
\end{array}$$

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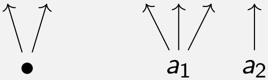
$\mathrm{Hom}(y^B, \mathbf{P}) \cong \mathbf{P}(B)$

Graphical



Combinatorial

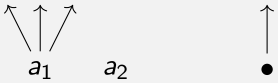
$$\begin{array}{ccccc} B & \xleftarrow{\mathbf{f}'} & \mathbf{P}'_a & \longrightarrow & \mathbf{P}' \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ 1 & = & 1 & \xrightarrow[a]{} & \mathbf{P}_0 \end{array}$$

$$\mathrm{Hom}(y^B, \mathbf{P}) \quad \cong \quad \sum_{a \in \mathbf{P}_0} B^{\mathbf{P}'_a} \quad \cong \quad \mathbf{P}(B)$$

Transforms: sections

$\mathrm{Hom}(\mathbf{P}, y) \cong \Gamma(\mathbf{P})$

Graphical



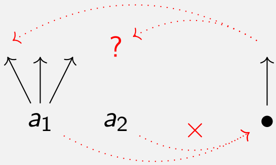
Combinatorial

$$\begin{array}{ccccc} \mathbf{P}' & \xleftarrow{\mathbf{f}'} & \mathbf{P}_0 & \longrightarrow & 1 \\ \downarrow & & \| & \lrcorner & \| \\ \mathbf{P}_0 & = & \mathbf{P}_0 & \xrightarrow[!]{} & 1 \end{array}$$

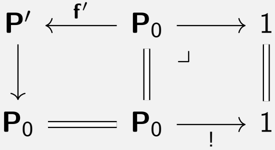
Transforms: sections

$\mathrm{Hom}(\mathbf{P}, y) \cong \Gamma(\mathbf{P})$

Graphical



Combinatorial



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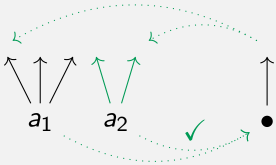
Graphical



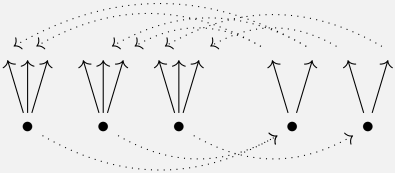
Combinatorial

$$\begin{array}{ccccc} \mathbf{P}' & \xleftarrow{\mathbf{f}'} & \mathbf{P}_0 & \longrightarrow & 1 \\ \downarrow & & \| & \lrcorner & \| \\ \mathbf{P}_0 & = & \mathbf{P}_0 & \xrightarrow[!]{} & 1 \end{array}$$

Transforms: lenses

$\mathbf{P} = Sy^S$ (server) $\qquad\qquad$ $\mathbf{Q} = Cy^C$ (client)

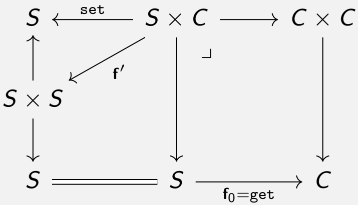
Graphical



Algebraic

$$\begin{array}{ccc} S \times X^S & \xrightarrow{\langle \mathbf{f}_0, - \circ \mathbf{f}'_s \rangle} & C \times X^C \\ {\scriptstyle S \times k \circ -}\big\downarrow & & \big\downarrow{\scriptstyle C \times k \circ -} \\ S \times Y^S & \xrightarrow[\langle \mathbf{f}_0, - \circ \mathbf{f}'_s \rangle]{} & C \times Y^C \end{array}$$

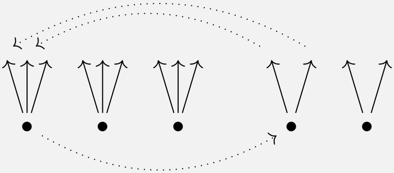
Combinatorial



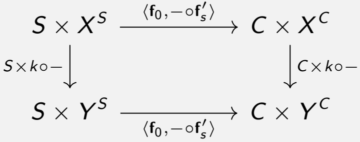
Transforms: lenses

$\mathbf{P} = Sy^S$ (server) $\qquad$ $\mathbf{Q} = Cy^C$ (client)

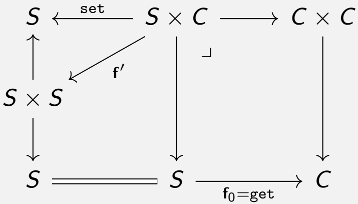
Graphical



Algebraic



Combinatorial



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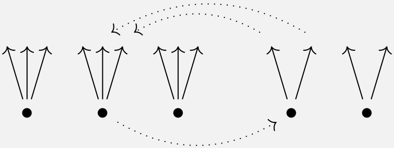
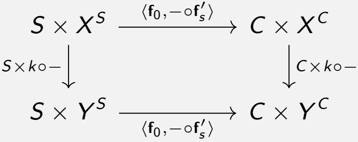
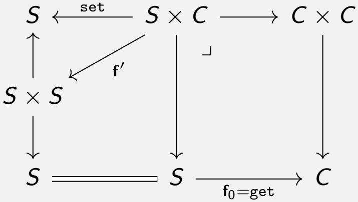
Graphical



Algebraic



Combinatorial



Transforms: lenses

$\mathbf{P} = Sy^S$ (server)

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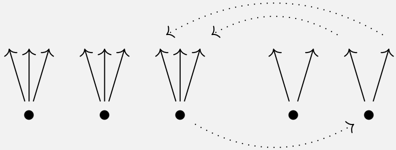
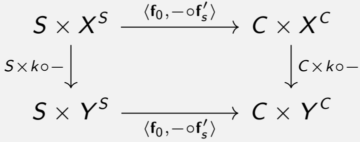
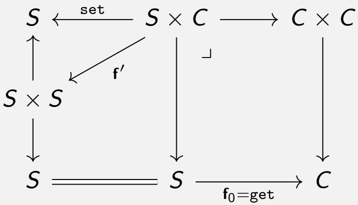
Graphical



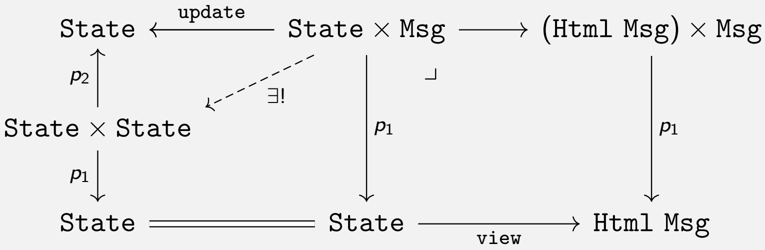
Algebraic



Combinatorial



The Elm loop



State
update
State × Msg
(Html Msg) × Msg
p2
∃!
State × State
p1
p1
p1
State
State
view
Html Msg

Semagrams

Demo

Monads

"A monad is just™ a monoid in the category of endofunctors, what's the problem?"

- (not) Philip Wadler (cf. James Iry)

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`return`	`join`
$y \to M$	$M \lhd M \to M$

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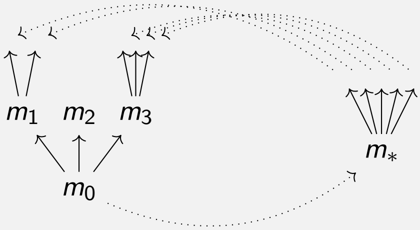
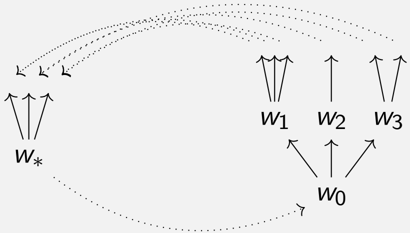


Comonads

"A comonad is just™ a comonoid in the category of endofunctors, what's the problem?"

- (not) Philip Wadler (not cf. James Iry)

`extract`	`dup`
$W \to y$	$W \to W \triangleleft W$

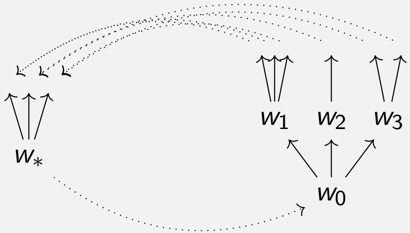


Comonads

"A category is just™ a polynomial comonoid in the category of endofunctors, what's the problem?"

- (not) Ahman & Uustalu (not cf. James Iry)

`extract`	`dup`
$W \to y$	$W \to W \triangleleft W$



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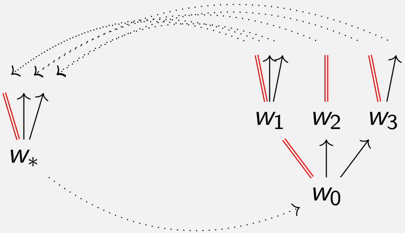
$W \to y$	$W \to W \lhd W$



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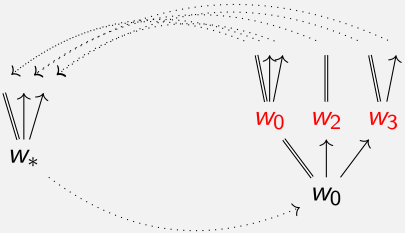
$W \to y$	$W \to W \triangleleft W$



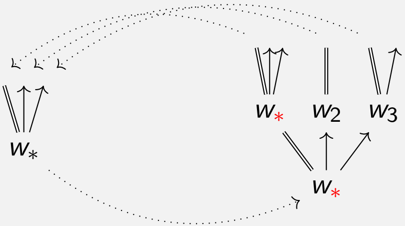
Comonads

"A category is just™ a polynomial comonoid in the category of endofunctors, what's the problem?"

- (not) Ahman & Uustalu (not cf. James Iry)

`extract` $W \to y$

`dup` $W \to W \triangleleft W$



$\times$ and $\otimes$

Monoidal products on $\mathbb{C}$ lift to monoidal products on $\mathbf{Set}^{\mathbb{C}}$

$$\mathbb{C} = (\mathbf{Set}, +)$$

$$\mathbf{P} \times \mathbf{Q} = \sum_{a \in \mathbf{P}_0} \sum_{b \in \mathbf{Q}_0} y^{\mathbf{P}'_a + \mathbf{Q}'_b}$$

$$\begin{array}{c} (\mathbf{P}' \times \mathbf{Q}_0) + (\mathbf{P}_0 \times \mathbf{Q}') \\ \downarrow \\ \mathbf{P}_0 \times \mathbf{Q}_0 \end{array}$$

$$\mathbb{C} = (\mathbf{Set}, \times)$$

$$\mathbf{P} \otimes \mathbf{Q} = \sum_{a \in \mathbf{P}_0} \sum_{b \in \mathbf{Q}_0} y^{\mathbf{P}'_a \times \mathbf{Q}'_b}$$

$$\begin{array}{c} \mathbf{P}' \times \mathbf{Q}' \\ \downarrow \\ \mathbf{P}_0 \times \mathbf{Q}_0 \end{array}$$

$\times$ and $\otimes$

$$Sy^S \xrightarrow{\quad \langle \mathbf{f},\mathbf{g}\rangle \quad} \mathbf{P}\times\mathbf{Q}$$

$$\begin{array}{ccccc}
S\times S & \xleftarrow{[\mathbf{f}',\mathbf{g}']} & \begin{matrix}\mathbf{P}'_{\mathbf{f}}\\ + \\ \mathbf{Q}'_{\mathbf{g}}\end{matrix} \;\lrcorner & \longrightarrow & \begin{matrix}\mathbf{P}'\times\mathbf{Q}_0\\ + \\ \mathbf{P}_0\times\mathbf{Q}'\end{matrix} \\
\downarrow & & \downarrow & & \downarrow \\
S & = & S & \xrightarrow[\langle \mathbf{f}_0,\mathbf{g}_0\rangle]{} & \mathbf{P}_0\times\mathbf{Q}_0
\end{array}$$

$\times$ and $\otimes$

$$Sy^S \xrightarrow{\langle \mathbf{f},\mathbf{g}\rangle} \mathbf{P} \times \mathbf{Q}$$

$$\begin{array}{ccccc}
S \times S & \xleftarrow{[\mathbf{f}',\mathbf{g}']} & \begin{array}{c}\mathbf{P}'_{\mathbf{f}} \\ + \\ \mathbf{Q}'_{\mathbf{g}}\end{array} \lrcorner & \longrightarrow & \begin{array}{c}\mathbf{P}' \times \mathbf{Q}_0 \\ + \\ \mathbf{P}_0 \times \mathbf{Q}'\end{array} \\
\downarrow & & \downarrow & & \downarrow \\
S & = & S & \xrightarrow[\langle \mathbf{f}_0,\mathbf{g}_0\rangle]{} & \mathbf{P}_0 \times \mathbf{Q}_0
\end{array}$$

"Independent interfaces"

$\times$ and $\otimes$

$$Sy^S \xrightarrow{\quad\mathbf{f}\quad} \mathbf{P} \otimes \mathbf{Q}$$

$$\begin{array}{ccccc} S \times S & \xleftarrow{\;\mathbf{f}'\;} & \mathbf{P}'_g \times \mathbf{Q}'_h & \longrightarrow & \mathbf{P}' \times \mathbf{Q}' \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ S & = & S & \xrightarrow[\mathbf{f}_0=\langle g,h\rangle]{} & \mathbf{P}_0 \times \mathbf{Q}_0 \end{array}$$

$\times$ and $\otimes$

$$Sy^S \xrightarrow{\quad \mathbf{f} \quad} \mathbf{P} \otimes \mathbf{Q}$$

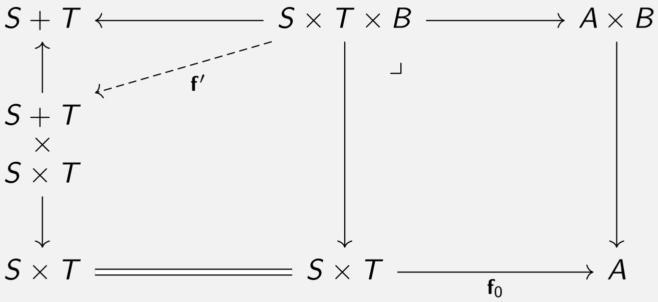
$$\begin{array}{ccccc}
S \times S & \xleftarrow{\;\mathbf{f}'\;} & \mathbf{P}'_g \times \mathbf{Q}'_h & \longrightarrow & \mathbf{P}' \times \mathbf{Q}' \\
\downarrow & & \downarrow & \lrcorner & \downarrow \\
S & = & S & \xrightarrow[\mathbf{f}_0 = \langle g, h \rangle]{} & \mathbf{P}_0 \times \mathbf{Q}_0
\end{array}$$

"Simultaneous interfaces"

(e.g., keyboard & mouse)

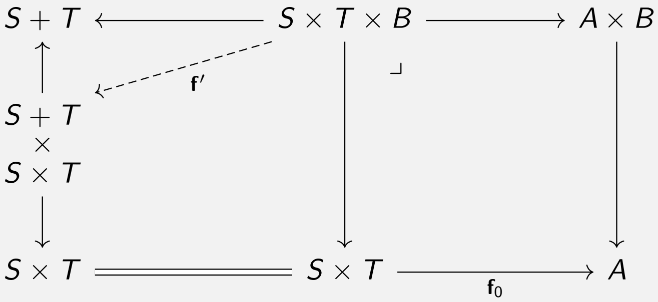
$\times$ and $\otimes$

$$Sy^S \times Ty^T \xrightarrow{\mathbf{f}} Ay^B$$



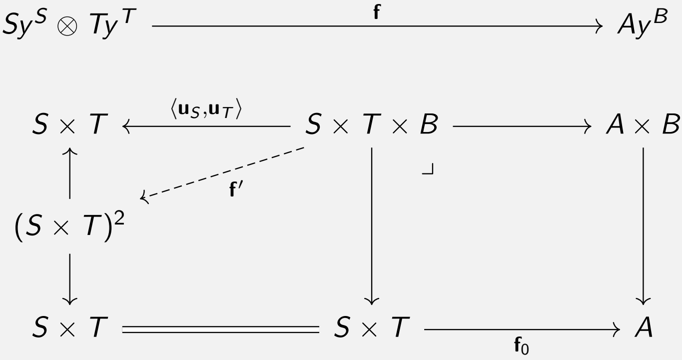
$\times$ and $\otimes$

$$Sy^S \times Ty^T \xrightarrow{\mathbf{f}} Ay^B$$

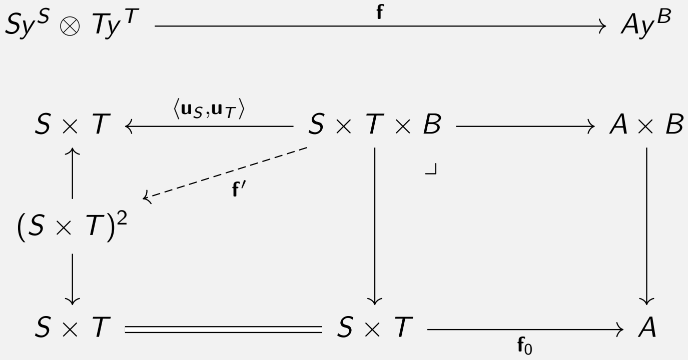


"Interacting components (but...)"

$\times$ and $\otimes$

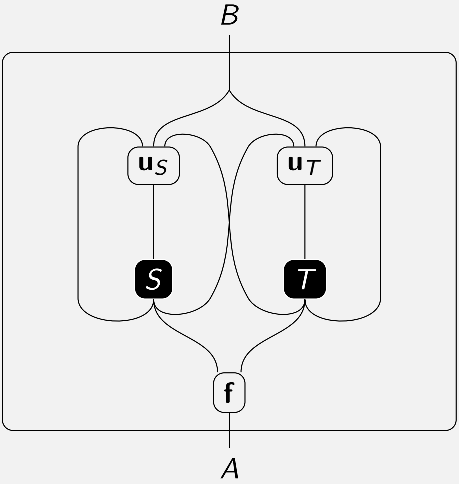


$\times$ and $\otimes$



"Interacting components"

Interacting components



Monoidal closure(s)

$$\mathbf{R}^{\mathbf{Q}} = \prod_{b \in \mathbf{Q}_0} \mathbf{R}(y + \mathbf{Q}'_b) \qquad [\mathbf{Q}, \mathbf{R}] = \prod_{b \in \mathbf{Q}_0} \mathbf{R}(y \times \mathbf{Q}'_b)$$

Monoidal closure(s)

$\mathbf{P} \times \mathbf{Q} \to \mathbf{R}$

$\mathbf{P} \otimes \mathbf{Q} \to \mathbf{R}$

Monoidal closure(s)

$$\frac{\mathbf{P} \times \mathbf{Q} \to \mathbf{R}}{\sum_{a,b} y^{\mathbf{P}'_a + \mathbf{Q}'_b} \quad \to \quad \mathbf{R}}$$

$$\frac{\mathbf{P} \otimes \mathbf{Q} \to \mathbf{R}}{\sum_{a,b} y^{\mathbf{P}'_a \times \mathbf{Q}'_b} \quad \to \quad \mathbf{R}}$$

Monoidal closure(s)

$$\frac{\dfrac{\mathbf{P} \times \mathbf{Q} \to \mathbf{R}}{\sum_{a,b} y^{\mathbf{P}'_a + \mathbf{Q}'_b} \quad \to \quad \mathbf{R}}}{\forall a, b \quad y^{\mathbf{P}'_a + \mathbf{Q}'_b} \to \mathbf{R}}$$

$$\frac{\dfrac{\mathbf{P} \otimes \mathbf{Q} \to \mathbf{R}}{\sum_{a,b} y^{\mathbf{P}'_a \times \mathbf{Q}'_b} \quad \to \quad \mathbf{R}}}{\forall a, b \quad y^{\mathbf{P}'_a \times \mathbf{Q}'_b} \to \mathbf{R}}$$

Monoidal closure(s)

$$\frac{\frac{\frac{\mathbf{P} \times \mathbf{Q} \to \mathbf{R}}{\sum_{a,b} y^{\mathbf{P}'_a + \mathbf{Q}'_b} \quad \to \quad \mathbf{R}}}{\forall a, b \quad y^{\mathbf{P}'_a + \mathbf{Q}'_b} \to \mathbf{R}}}{\forall a, b \quad \mathbf{R}(\mathbf{P}'_a + \mathbf{Q}'_b)}$$

$$\frac{\frac{\frac{\mathbf{P} \otimes \mathbf{Q} \to \mathbf{R}}{\sum_{a,b} y^{\mathbf{P}'_a \times \mathbf{Q}'_b} \quad \to \quad \mathbf{R}}}{\forall a, b \quad y^{\mathbf{P}'_a \times \mathbf{Q}'_b} \to \mathbf{R}}}{\forall a, b \quad \mathbf{R}(\mathbf{P}'_a \times \mathbf{Q}'_b)}$$

Monoidal closure(s)

$\mathbf{P} \times \mathbf{Q} \to \mathbf{R}$	$\mathbf{P} \otimes \mathbf{Q} \to \mathbf{R}$
$\sum_{a,b} y^{\mathbf{P}'_a + \mathbf{Q}'_b} \quad \to \quad \mathbf{R}$	$\sum_{a,b} y^{\mathbf{P}'_a \times \mathbf{Q}'_b} \quad \to \quad \mathbf{R}$
$\forall a, b \quad y^{\mathbf{P}'_a + \mathbf{Q}'_b} \to \mathbf{R}$	$\forall a, b \quad y^{\mathbf{P}'_a \times \mathbf{Q}'_b} \to \mathbf{R}$
$\forall a, b \quad \mathbf{R}(\mathbf{P}'_a + \mathbf{Q}'_b)$	$\forall a, b \quad \mathbf{R}(\mathbf{P}'_a \times \mathbf{Q}'_b)$
$\forall a, b \quad y^{\mathbf{P}'_a} \to \mathbf{R}(y + \mathbf{Q}'_b)$	$\forall a, b \quad y^{\mathbf{P}'_a} \to \mathbf{R}(y \times \mathbf{Q}'_b)$

Monoidal closure(s)

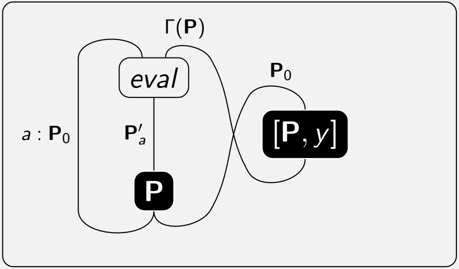
$$\frac{\frac{\frac{\frac{\frac{\frac{\mathbf{P} \times \mathbf{Q} \to \mathbf{R}}{\sum_{a,b} y^{\mathbf{P}'_a + \mathbf{Q}'_b} \quad \to \quad \mathbf{R}}}{\forall a, b \quad y^{\mathbf{P}'_a + \mathbf{Q}'_b} \to \mathbf{R}}}{\forall a, b \quad \mathbf{R}(\mathbf{P}'_a + \mathbf{Q}'_b)}}{\forall a, b \quad y^{\mathbf{P}'_a} \to \mathbf{R}(y + \mathbf{Q}'_b)}}{\sum_a y^{\mathbf{P}'_a} \to \prod_b \mathbf{R}(y + \mathbf{Q}'_b)}}{\mathbf{P} \to \mathbf{R}^{\mathbf{Q}}}$$

$$\frac{\frac{\frac{\frac{\frac{\frac{\mathbf{P} \otimes \mathbf{Q} \to \mathbf{R}}{\sum_{a,b} y^{\mathbf{P}'_a \times \mathbf{Q}'_b} \quad \to \quad \mathbf{R}}}{\forall a, b \quad y^{\mathbf{P}'_a \times \mathbf{Q}'_b} \to \mathbf{R}}}{\forall a, b \quad \mathbf{R}(\mathbf{P}'_a \times \mathbf{Q}'_b)}}{\forall a, b \quad y^{\mathbf{P}'_a} \to \mathbf{R}(y \times \mathbf{Q}'_b)}}{\sum_a y^{\mathbf{P}'_a} \to \prod_b \mathbf{R}(y \times \mathbf{Q}'_b)}}{\mathbf{P} \to \prod_b [\mathbf{Q}, \mathbf{R}]}$$

Monoidal closure

Something (universal) in a box with **P**

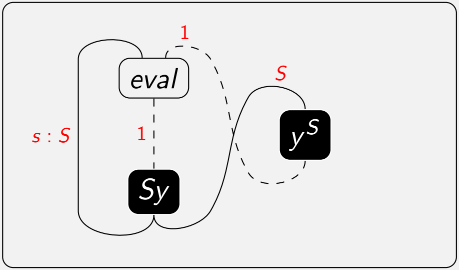
$$[\mathbf{P}, y] \cong \prod_{a \in \mathbf{P}_0} y \times \mathbf{P}'_a \cong \Gamma(\mathbf{P}) y^{\mathbf{P}_0}$$



Monoidal closure

Something (universal) in a box with Sy (reader)

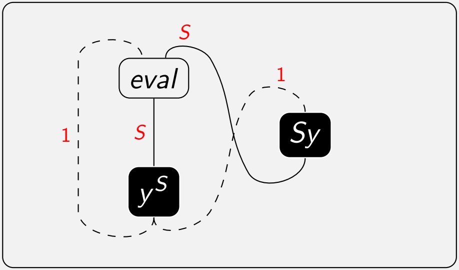
$$[Sy, y] \cong \prod_{s \in S} y \times 1 \cong y^S$$



Monoidal closure

Something (universal) in a box with y^S (writer)

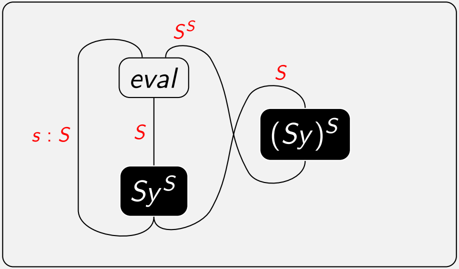
$$[y^S, y] \cong \prod_{a \in 1} y \times S \cong Sy$$



Monoidal closure

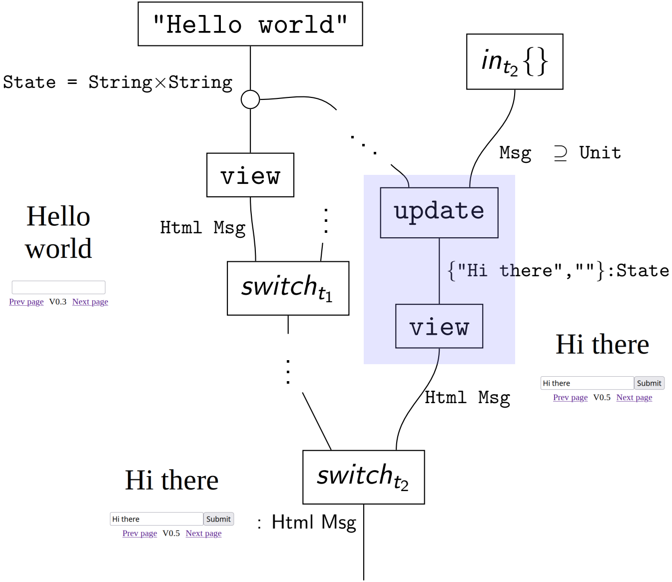
Something (universal) in a box with Sy^S (store)

$$[Sy^S, y] \cong \prod_{s \in \S} y \times S \cong (Sy)^S$$

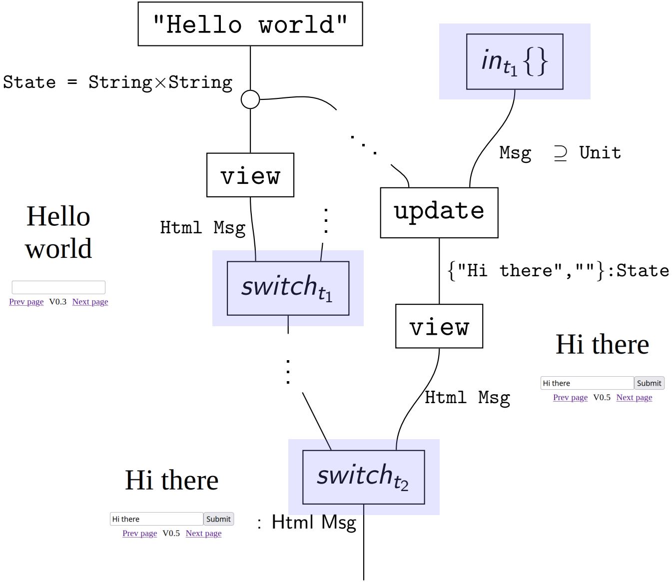


Other topics

Functional Reactive Programming

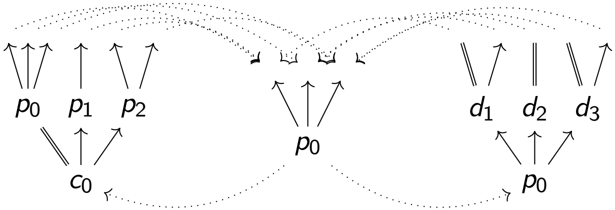


Functional Reactive Programming



Bicomodules

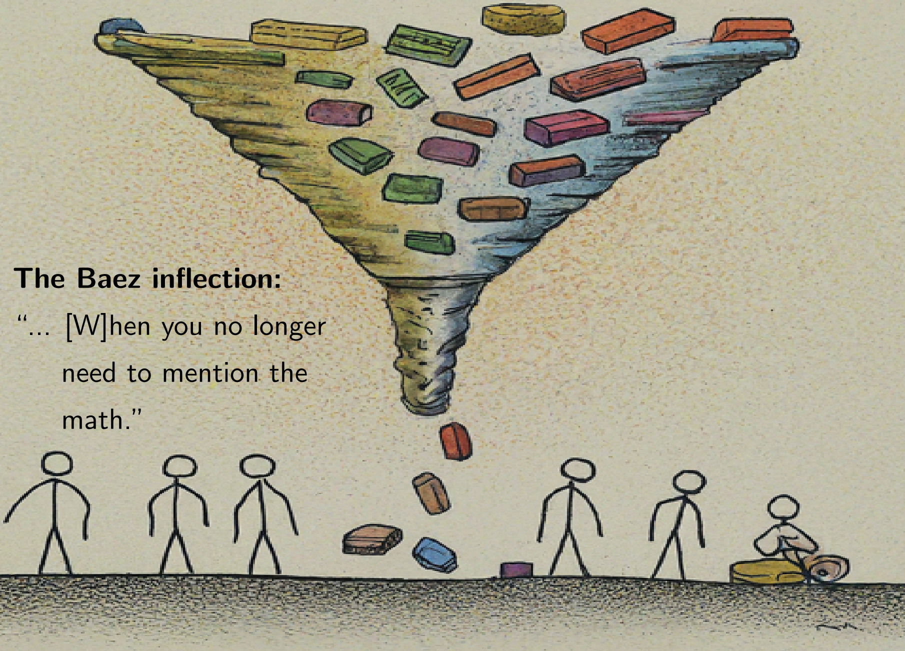
$$\mathbb{C} \triangleleft \mathbf{P} \longleftarrow \mathbf{P} \longrightarrow \mathbf{P} \triangleleft \mathbb{D}$$



What is category theory for?

The Baez inflection:

"… [W]hen you no longer need to mention the math."



Bartosz' Principle: Category theory is for humans

"Category theory is a very good description of how our minds work."

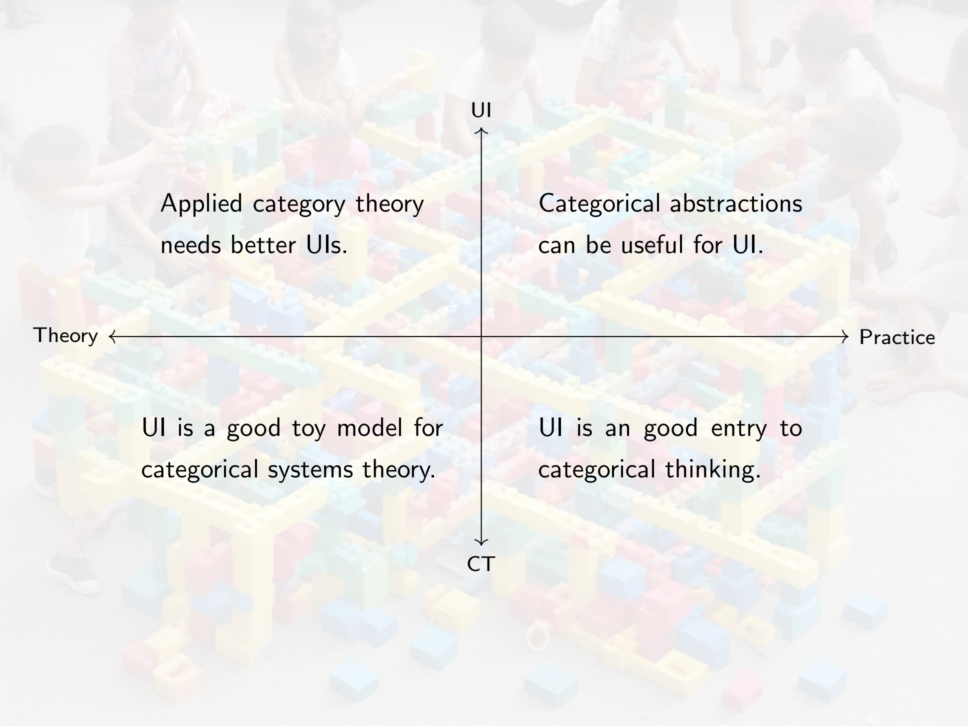
Cheng's Theorem:

Mathematics is what is easy*

Category theory is the mathematics of mathematics

Category theory is what is easy* about what is easy*

*May require "logical thought processes"


UI
Applied category theory needs better UIs.
Categorical abstractions can be useful for UI.
Theory
Practice
UI is a good toy model for categorical systems theory.
UI is an good entry to categorical thinking.
CT

UI

Applied category theory needs better UIs.

Categorical abstractions can be useful for UI.

Theory — Semagrams (some day...) — Practice

UI is a good toy model for categorical systems theory.

UI is an good entry to categorical thinking.

CT

References