

# Kleisli Constructions For Pseudomonads

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- Enriched Kleisli objects  
for pseudomonads

arxiv: 2311.15618

- Tricategorical Universal  
properties via enriched  
homotopy theory

arxiv: 2409.01837

# Plan

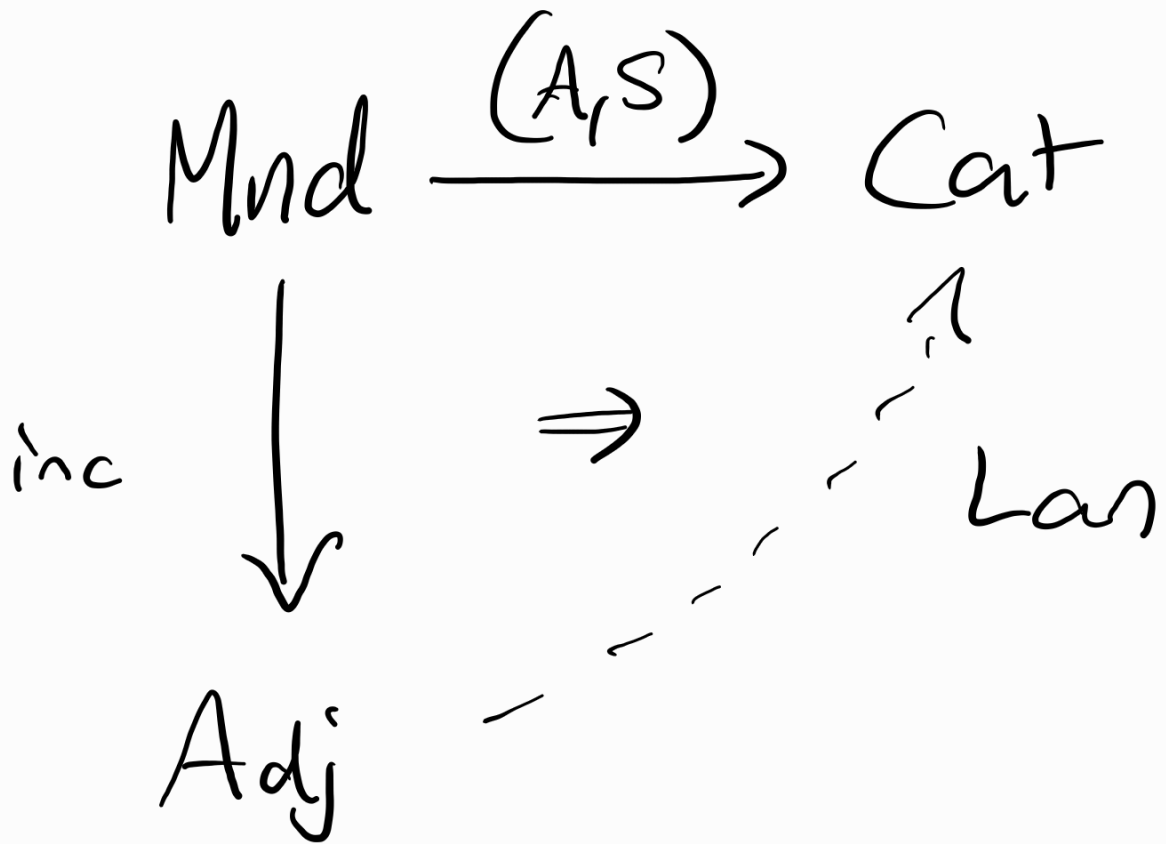
1. Review Kleisli for monads.
2. Describe Kleisli for pseudomonads.
3. Examine nice homotopical properties of the enriched Kleisli object.
4. Discuss Tricategorical Universal Properties.
5. Further Work / Applications

# 1. Kleisli for monads on categories (Abstractly)

Given  $(A, S)$  monad

want 
$$\begin{array}{ccc} & A & \\ F_S \downarrow & \dashv & \uparrow U_S \\ & A_S & \end{array}$$
 adjunction splitting  $(A, S)$

$$\begin{array}{ccc} & A & \\ F_S \swarrow & & \searrow F + U \\ A_S & \dashv \vdash & B \end{array}$$
 splitting  $(A, S)$



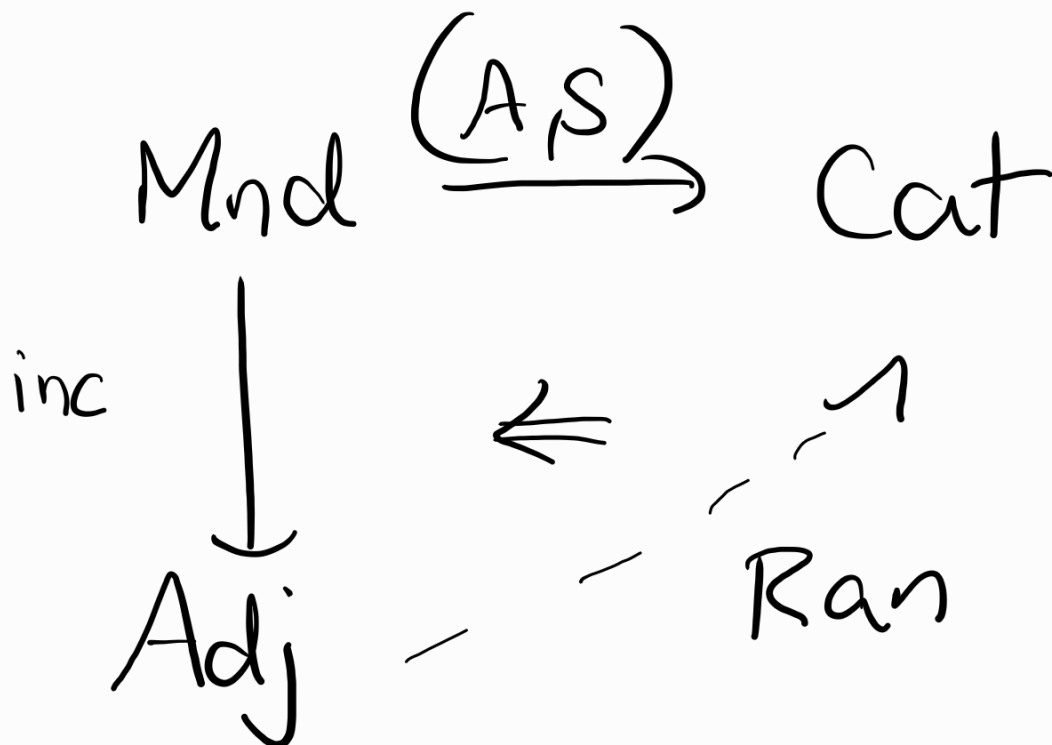
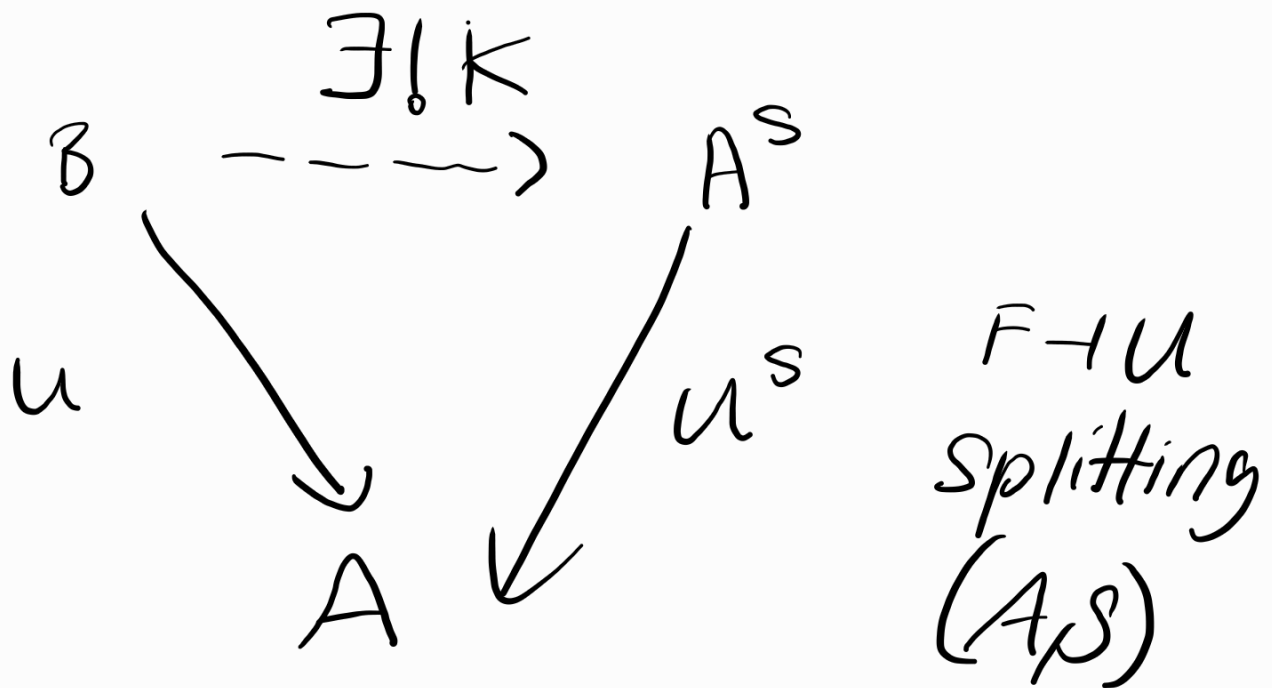
co limit of  $\text{Mnd} \xrightarrow{(A, S)} \text{Cat}$

weighted by

$$\text{Mnd}^{\text{op}} \longrightarrow \text{Cat}$$

$$\left( \Delta_+, -\oplus 1 \right)$$

# Eilenberg-Moore for monads on categories



$A^S$  has objects

$$(X, \phi: SX \rightarrow X)$$

satisfying

$$\begin{array}{ccc}
 X \xrightarrow{1_X} SX & & S^2X \xrightarrow{S\phi} SX \\
 \downarrow \phi & & \downarrow \phi \\
 X & & X
 \end{array}$$

$\&$  morphisms  $(X, \phi) \xrightarrow{f} (Y, \psi)$

given by  $X \xrightarrow{f} Y$  satisfying

$$\begin{array}{ccc}
 SX & \xrightarrow{Sf} & SY \\
 \downarrow \phi & & \downarrow \psi \\
 X & \xrightarrow{f} & Y
 \end{array}$$

# 1. Kleisli for monads on categories (concretely)

$$\text{ob}(A_S) = \text{ob}(A) \quad S$$

and ...

1. ... morphisms  $X \rightarrow Y$

•  $f \in A(X, SY)$

• identities  $X \xrightarrow{\eta_X} SX$

• composition  $g \circ_S f \circ_S =$

$$X \xrightarrow{f} SY \xrightarrow{Sg} S^2Z \xrightarrow{\mu_Z} SZ$$

2. ... morphisms

$X \xrightarrow{f} Y$  given by

morphisms of algebras

$$(S_X, \mu_X) \longrightarrow (S_Y, \mu_Y)$$

3. ... morphisms generated

by those in  $A$  +

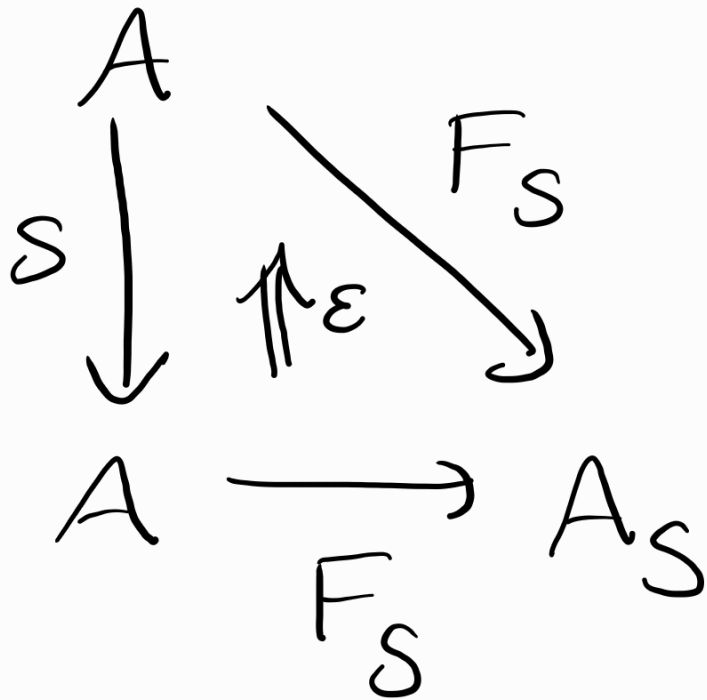
formal algebra

structures  $SX \xrightarrow{\varepsilon_X} X$

respected by all other morphisms.



Description 3 in more  
detail:



such that  $F_S \circ s \xrightarrow{\varepsilon} F_S$

is an algebra for

$$- \circ s : [A, A_S] \rightarrow [A, A_S]$$

# Generators

$$f \in A(X, Y)$$

---

$$f: X \rightarrow Y \quad \text{gen morphism}$$

$$X \in A$$

---

$$SX \xrightarrow{\epsilon_X} X$$

gen. morphism

# Relations

$$X \xrightarrow{\eta_X} SX$$

$$\begin{array}{ccc} & & \downarrow \epsilon_X \\ & \searrow \tau_X & X \end{array}$$

$$S^2X \xrightarrow{\epsilon_{SX}} SX$$

$$\begin{array}{ccc} \downarrow \mu_X & & \downarrow \epsilon_X \\ SX & \xrightarrow{\epsilon_X} & SX \end{array}$$

$$SX \xrightarrow{Sf} SY$$

$$\begin{array}{ccc} \downarrow \epsilon_X & & \downarrow \epsilon_Y \\ X & \xrightarrow{f} & Y \end{array}$$

# The isomorphism of categories

{ gen.  
+ relations }

$$X \xrightarrow{f} Y$$



$$X \xrightarrow{f} Y \xrightarrow{\eta_Y} SY$$

$$SX \xrightarrow{\epsilon_X} X$$



$$SX \xrightarrow{1_{SX}} SX$$



{ Kleisli maps  
& composition }

$$X \xrightarrow{f} SY \xrightarrow{\varepsilon_Y} Y$$



$$X \xrightarrow{f} SY$$

## 2. Kleisli for Pseudomonads

Gray = { 2-categories  
2-functors  
pseudonat. transf.  
modifications

•  $A$  a 2-category

$$f \circ 1_x \not\cong f$$

$$(hg) \circ f \not\cong h(gf)$$

$$1_y \circ f \not\cong f$$

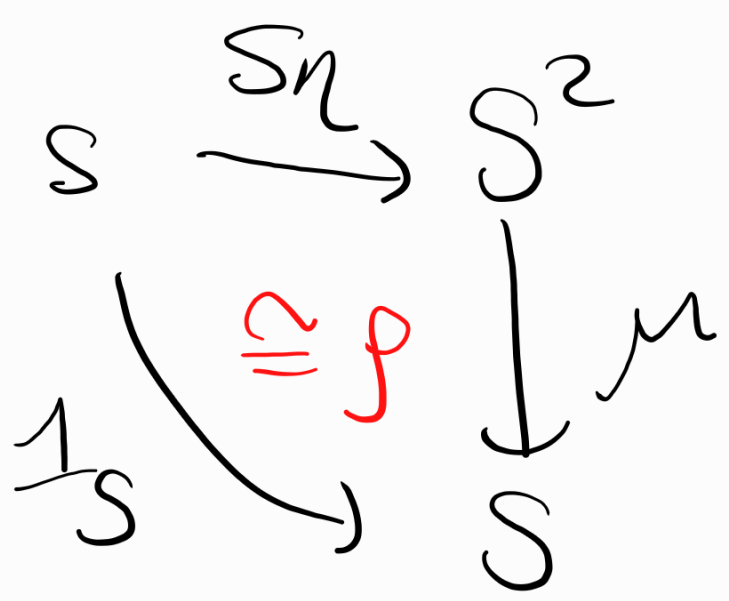
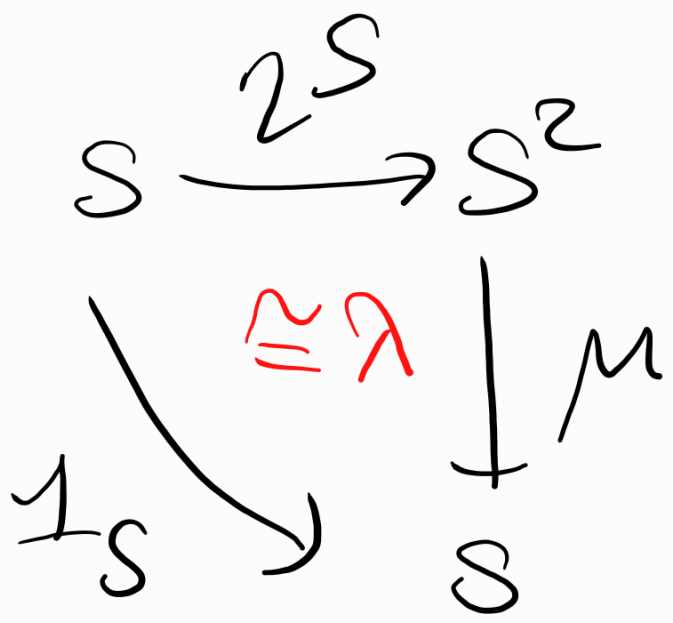
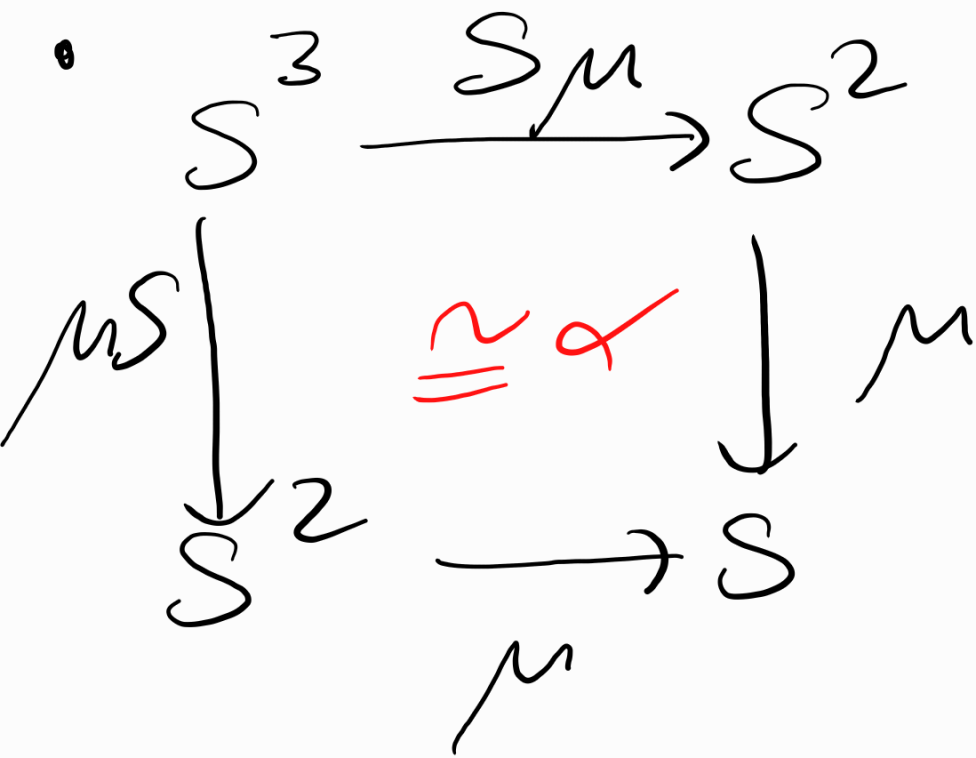
$S: A \rightarrow A$  a 2-functor

$$Sg Sf \neq S(gf)$$

$$1_{Sx} \neq S 1_x$$

$\eta, \mu$  pseudonatural transformations

$$\begin{array}{ccc} X \xrightarrow{f} Y & & S^2 X \xrightarrow{Sf} S^2 Y \\ \eta_x \downarrow \cong \mu_x & & \downarrow \eta_y \\ Sx \xrightarrow{Sf} Sy & & Sx \xrightarrow{Sf} Sy \end{array}$$



invertible modification

+ 2 axioms

(Marzouk 1997)

• (Lack 1999) Eilenberg-

Moore object for pseudo-

monads described as

a weighted limit

$\equiv$  { pseudoalgebras,  
pseudomorphisms  
ps. algebra  
transformations



(Cheng, Hyland, Power  
2003)

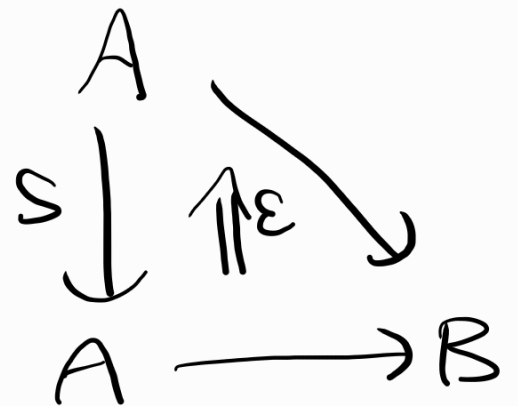
- Defined Kleisli bicategory

ie analogue of

$X \xrightarrow{f} SY$  + Kleisli composition

- Showed  $\text{Bicat}(A_S, B)$

bicquivalent to



"pseudoalgebras."

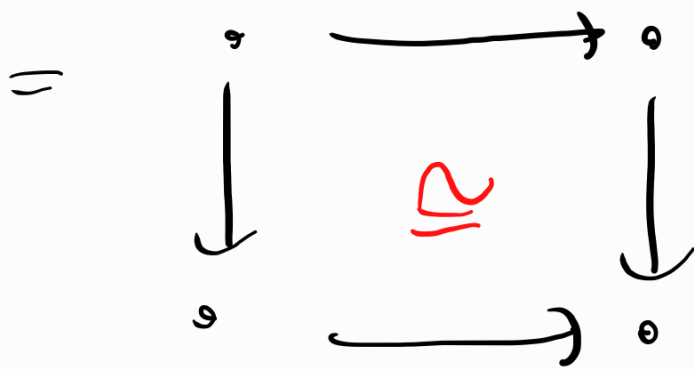
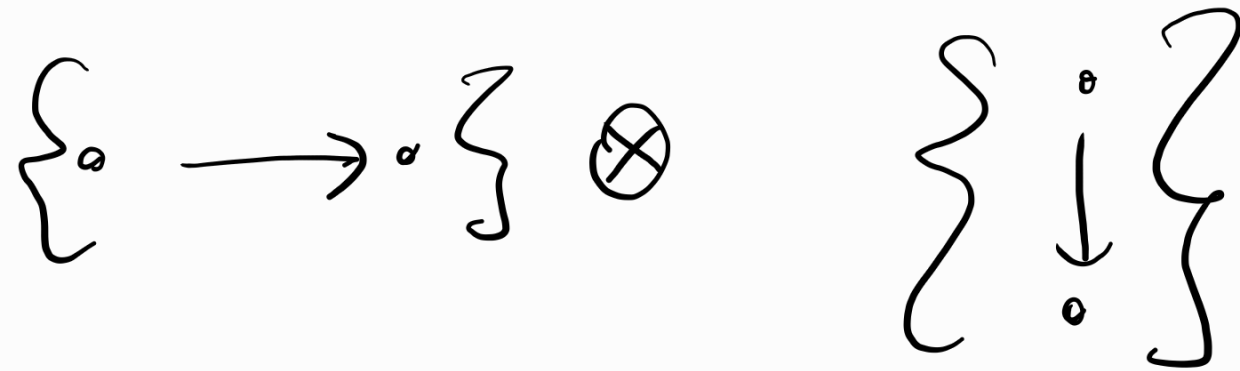
This can't be a  
Gray-weighted colimit  
as it is a bicategory!

(Gambino, Lobbia 2021)

Observed that free  
pseudalgebras are also  
not the enriched colimit  
as the comparison is only  
a biequivalence!

$$\text{Gray}(A_S, B) \longrightarrow \text{Gray}(A, B)^{\text{Gray}(S, B)}$$

$$\mathcal{K} = (2\text{-Cat}, \otimes, 1)$$



has internal hom

$$\text{Gray}(A, B) = \left\{ \begin{array}{l} \text{2-functors} \\ \text{ps. nat. transf.} \\ \text{modifications} \end{array} \right\}$$

& is locally presentable,

→ we can

at least describe

colimits via generators

& relations

(as in 3<sup>rd</sup> description  
of Kleisli category.)

$$\text{ob}(A_S) = \text{ob}(A)$$

$$\underline{f \in A(X, Y)}$$

$$f: X \rightarrow Y \text{ generator}$$

$$\underline{X \in A_0}$$

$$S_X \xrightarrow{\varepsilon_X} X$$

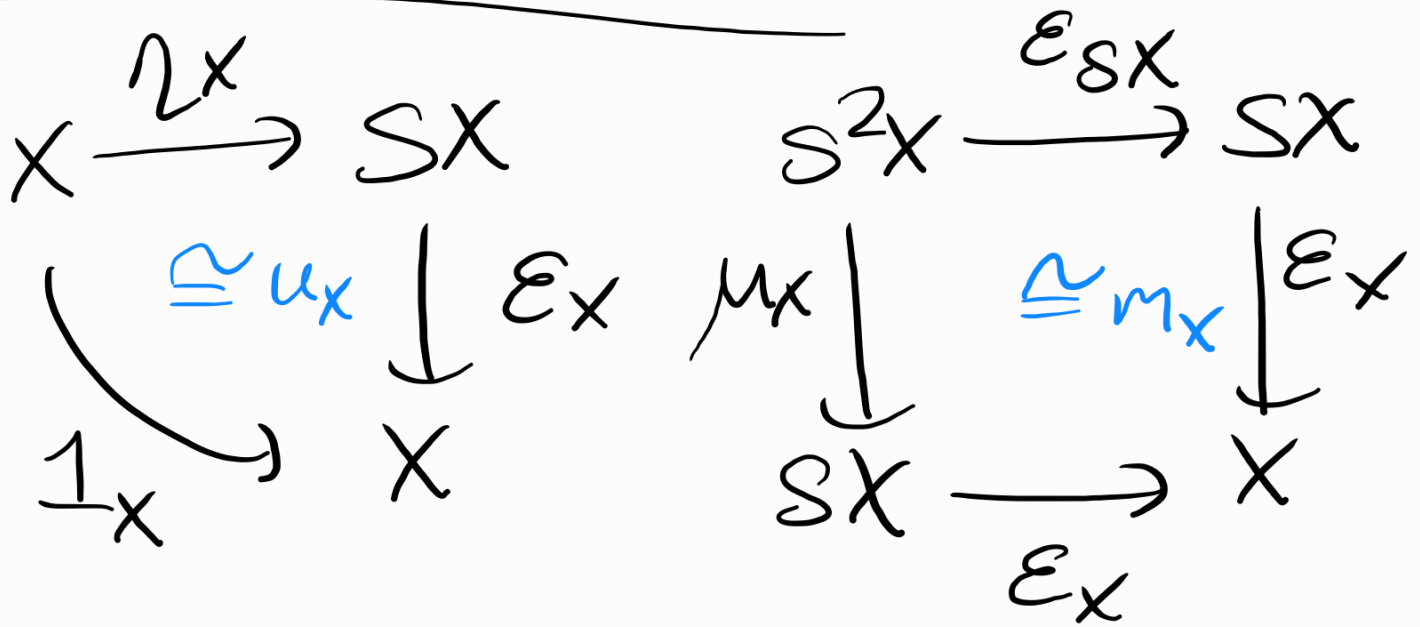
generator

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \parallel & \searrow & \\ \phi & & \\ \underbrace{\quad} & & \\ \vartheta & \xrightarrow{\quad} & \end{array} \in A$$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \parallel & \searrow & \\ \phi & & \\ \underbrace{\quad} & & \\ \vartheta & \xrightarrow{\quad} & \end{array} \text{gen} \\ \text{2-cell}$$

+ pseudoalgebra & pseudowrap

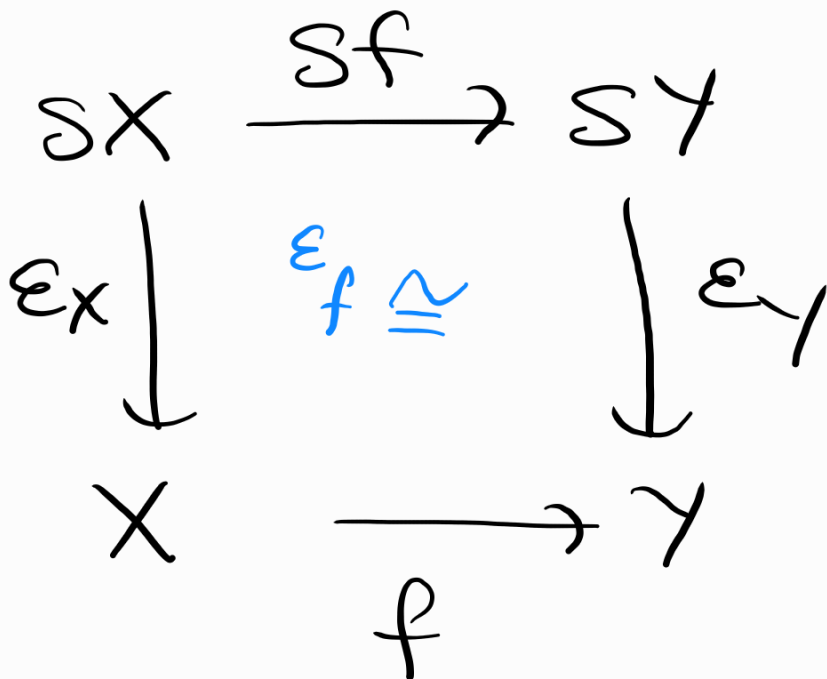
$X \in A$



gen.

2-cells.

$f \in A(X, Y)$

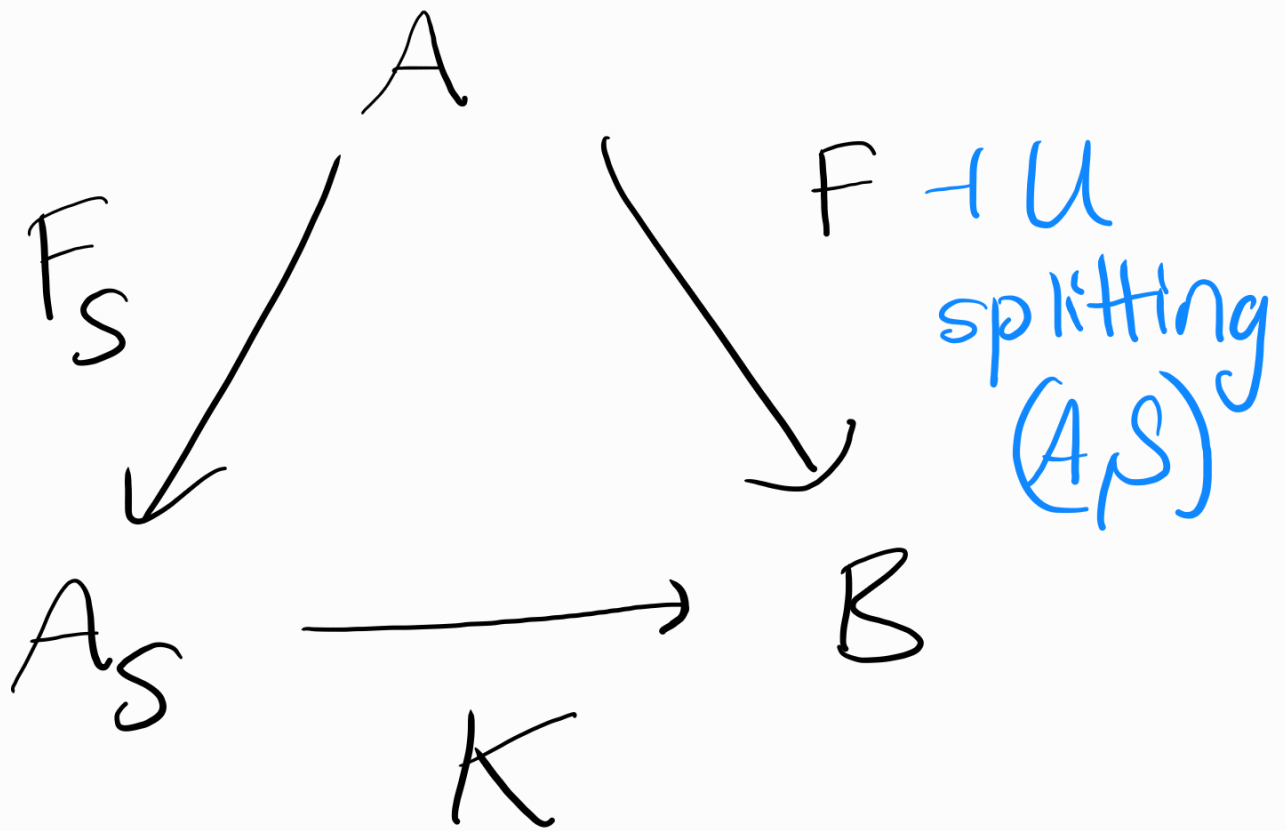


gen. 2-cell

Wism

relations.

Prop<sup>n</sup> (M. 2023)



is bi-fully faithful

ie  $A_S(X, Y) \longrightarrow B(KX, KY)$   
are equivalences of categories.

Th<sup>m</sup> (M. 2023)

The following are equivalent

1.  $F: A \rightarrow B$  is biess. surj  
on objects ie  $\forall Y \in B$

$$\exists X \in A \quad F X \simeq Y$$

2.  $K: A_S \rightarrow B$  is a

biequivalence ie biess  
surj + bi fully faithful

3. For all 2-categories  $C$

$\text{Gray}(B, C) \rightarrow \text{Gray}(A_S, C) \rightarrow \text{Gray}(A, C)^{\text{Gray}(B, C)}$   
is a biequivalence.

$\iff$  biequivalence.



# Example

$(1, \text{id}_1)$  has one obj.

one generating morphism

$$\varepsilon: * \rightarrow *$$

and two invertible gen.

2-cells

$$1_x \underset{u}{\overset{\sim}{\Rightarrow}} \varepsilon \underset{m}{\overset{\sim}{\Leftarrow}} \varepsilon^2$$

satisfying monad laws.

# Upshot

Enriched Kleisli left

pseudoadjoints are not  
closed under composition!

↳ Those satisfying the UP  
up to biequivalence  $\Leftrightarrow$

biess surj on objects

which are closed under  
composition.

### 3. Nice homotopical Properties

Q: How do we make  
precise these weaker  
Universal Properties?

A: Tricategorical limits &  
Colimits

Q: Won't that be  
very complicated?

A: Yes, but we can  
think of these as  
homotopy limits & colimits  
enriched over  
 $(\text{Gray}, \otimes, 1)$  with  
Lack's model structure!

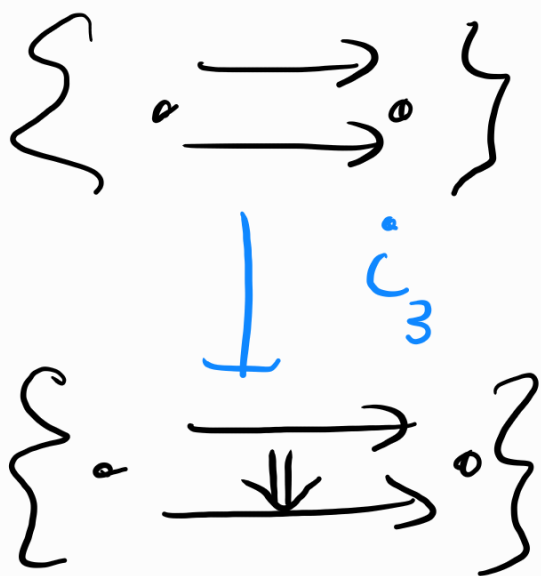
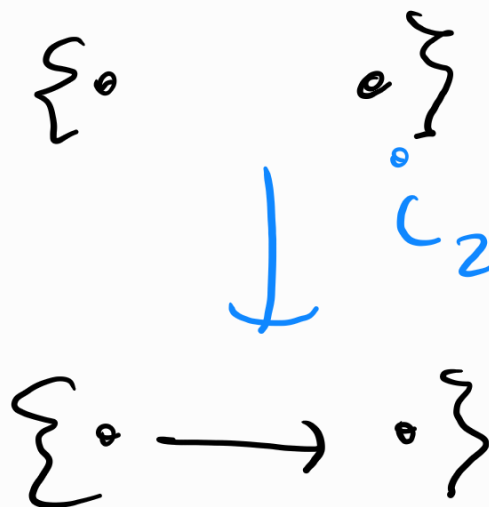
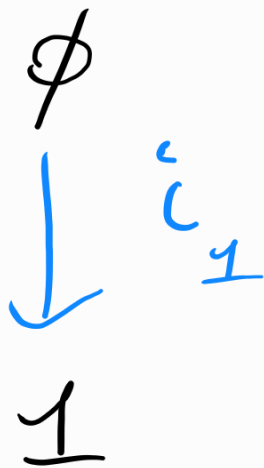
# Lack's model structure

fibrations:  
• lift adjoint eq.  
• lift isos in  
hom-categories

w.e : biequivalences.

cofibrant obj: 2-categories  
whose underlying category  
is free on a graph.

gen. cofibrations :



Th<sup>m</sup>

(M. 2023)

The weight for Kleisli  
objects is cofibrant

in the projective

model structure on

[Psmnd, Gray]

In fact, it can be built from the rep'ble

using pushouts &

copowers by  $i_2, i_3, i_4, \dots$

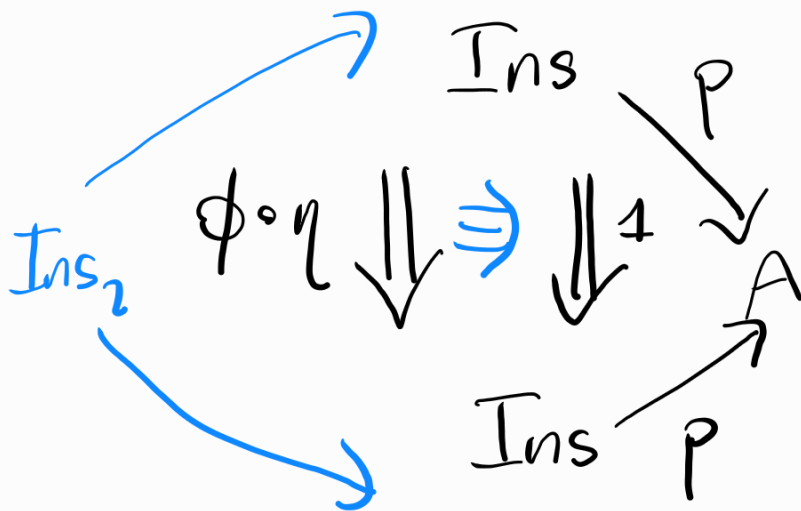
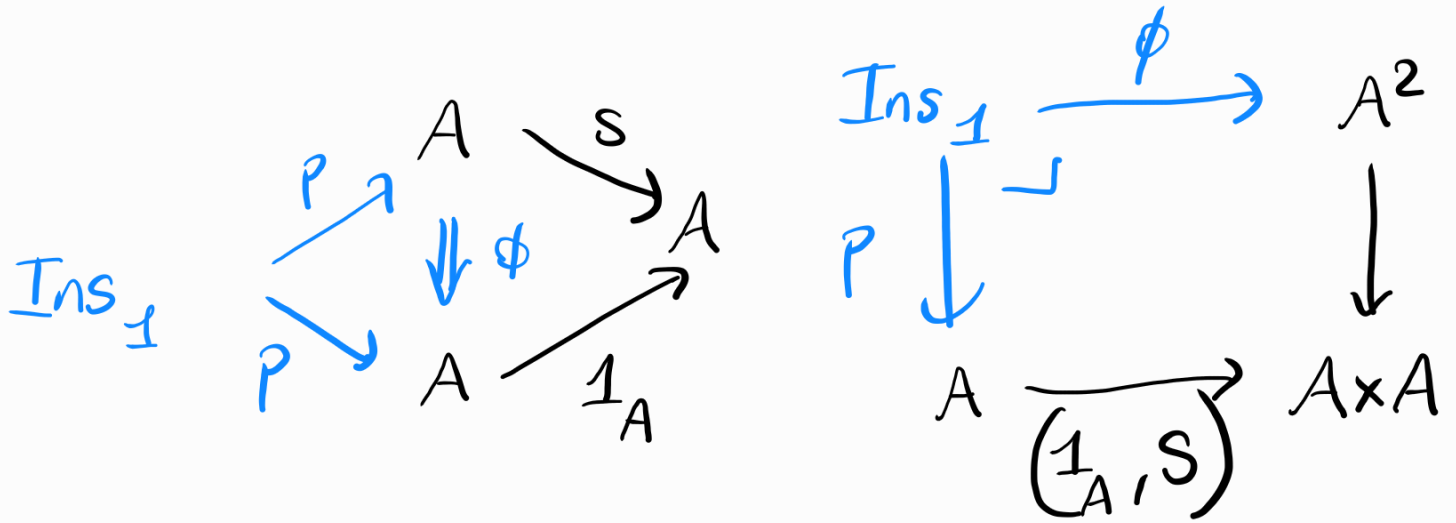
In particular,  $\begin{array}{c} \emptyset \\ \downarrow i_1 \\ 1 \end{array}$  &

retracts are not

needed.



# Construction of EM



# Dual construction of weight

$$\begin{array}{ccc}
 \text{Rep} + \text{Rep} \cong (1+1) \otimes \text{Rep} & \xrightarrow{i_2 \otimes \text{Rep}} & \{ \bullet \rightarrow \bullet \} \otimes \text{Rep} \\
 \downarrow (1, s) & & \downarrow \lrcorner \\
 \text{Rep} & \xrightarrow{\quad\quad\quad} & \text{Ins}_1
 \end{array}$$

$$\begin{array}{ccc}
 \{ \bullet \rightarrow \bullet \} \otimes \text{Ins}_1 & \xrightarrow{i_3 \otimes \text{Ins}_1} & \{ \bullet \xrightarrow{\mathbb{1}} \bullet \} \otimes \text{Ins}_1 \\
 \downarrow & & \downarrow \lrcorner \\
 \text{Ins}_1 & \xrightarrow{\quad\quad\quad} & \text{Ins}_2
 \end{array}$$

## 4. Tricategorical Universal Properties

$$A \xrightarrow{F} B$$

$$A^{\text{op}} \xrightarrow{W} \text{Bicat}$$

Def<sup>n</sup> (Power 2007)

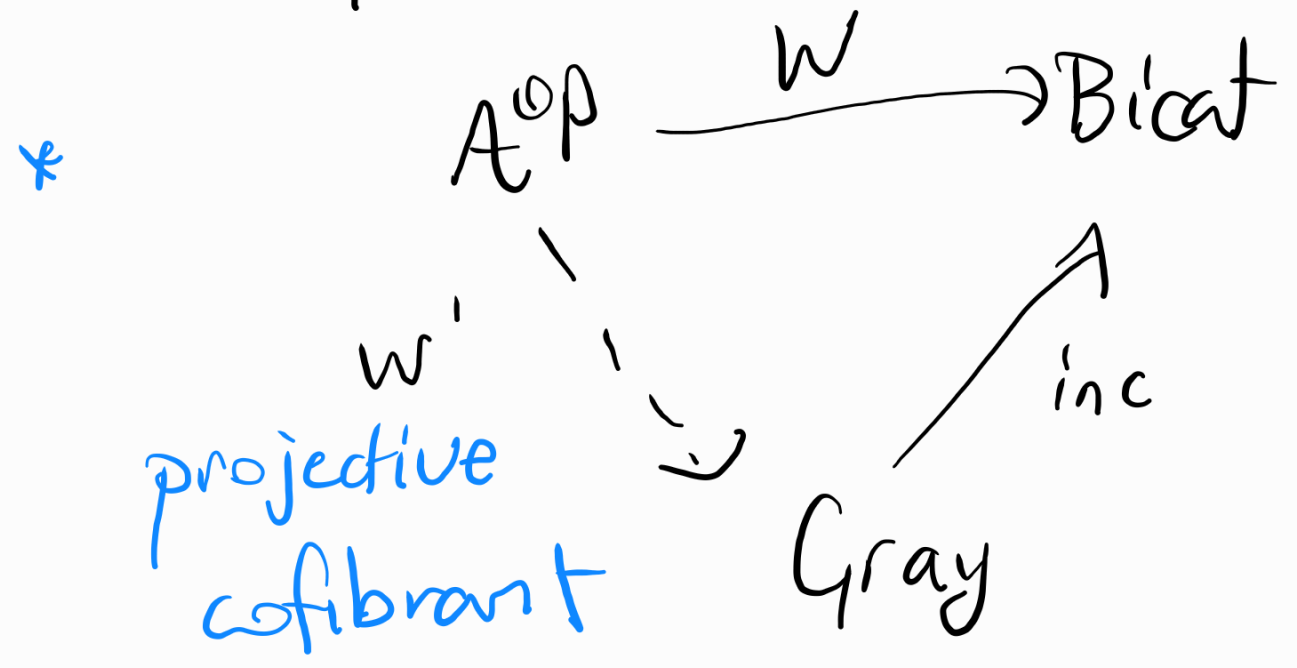
$$\text{Tricat}(A^{\text{op}}, \text{Bicat})(w, B(F, X))$$

$$\sim B(w \circ F, X)$$

In (Campbell 2016) & (M. 2023) these are

related to "nice" \*  
Gray enriched weighted  
colimits

\*  $A(X, Y)$  are cofibrant,



Th<sup>m</sup> (M. 2023)

Gray-

enriched Kleisli object of  
a pseudomonad is also

its tricategorical

colimit weighted by

$\text{Psmnd}^{\text{op}} \longrightarrow \text{Gray}$

$(\overline{\Delta}_+, -\oplus 1)$

Cor (M. 2023) The  
Kleisli bicategory &  
2-category of free  
pseudalgebras are

both also "trikleisli"

objects of the  
pseudomonad.

## 5. Applications / Further Work

- Free cocompletions for trikleisti objects / FTP<sub>B</sub>M<sub>2</sub>.

- 
- Symmetric / Sylleptic / Braided monoidal structures on pseudoalgebras / Kleisli bicategories.

# Status

- Explicit description given.
- VP of co-completion open.
- Applied to double-cats,  
(M. 2023) & to Categorical  
systems theory (Capucci, Myers  
2024)

- 
- Liftings & Extensions  
shown in arxiv 2402.11703  
& in an upcoming preprint.
  - Monoidal Kleisli treated by  
(Saville, Pacquet 2024)
  - tricategorical VP established.