Kleisli Constructions for Pseudomonads

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- Enriched Kleisli objects for pseudomonads
  arxiv: 2311.15618
- Tricategorical Universal properties via enriched homotopy theory
  arxiv: 2409.01837

Plan

1. Review Kleislis for monads.

2. Describe Kleislis for pseudomonads.

3. Examine nice homotopical properties of the enriched Kleisli object.

4. Discuss Tricategorical Universal Properties.

5. Further Work / Applications

1. Kleisli for monads on categories (Abstractly)

Given (A, S) monad

want

$$\begin{tikzcd} A \arrow[d, shift right=2, "F_S"'] \arrow[d, phantom, "\dashv"] \\ A_S \arrow[u, shift right=2, "U_S"'] \end{tikzcd}$$

adjunction splitting (A, S)

$$\begin{tikzcd} & A \arrow[dl, "F_S"'] \arrow[dr] & \\ A_S \arrow[rr, dashed, "\exists ! K"] & & B \end{tikzcd}$$

$F \dashv U$ splitting (A, S)

$$
\begin{tikzcd}
\mathrm{Mnd} \arrow[r, "{(A,S)}"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "\Rightarrow"] & \mathrm{Cat} \\
\mathrm{Adj} \arrow[ur, dashed, "\mathrm{Lan}"'] &
\end{tikzcd}
$$

colimit of $\mathrm{Mnd} \xrightarrow{(A,S)} \mathrm{Cat}$

weighted by

$$\mathrm{Mnd}^{op} \longrightarrow \mathrm{Cat}$$

$$(\Delta_+, - \oplus 1)$$

Eilenberg-Moore for monads on categories

$$\begin{tikzcd} B \arrow[rr, dashed, "\exists ! K"] \arrow[dr, "U"'] & & A^S \arrow[dl, "U^S"] \\ & A & \end{tikzcd}$$

$F \dashv U$ splitting (A,S)

$$\begin{tikzcd} \mathrm{Mnd} \arrow[rr, "(A,S)"] \arrow[d, "\mathrm{inc}"'] & & \mathrm{Cat} \\ \mathrm{Adj} \arrow[urr, dashed, "\mathrm{Ran}"'] \arrow[r, phantom, "\Leftarrow"] & & \end{tikzcd}$$

A^S has objects

$(X, \phi : SX \to X)$

satisfying

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, bend right, "1_X"'] & SX \arrow[d, "\phi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} S^2X \arrow[r, "S\phi"] \arrow[d, "\mu_X"'] & SX \arrow[d, "\phi"] \\ SX \arrow[r, "\phi"'] & X \end{tikzcd}$$

& morphisms $(X, \phi) \xrightarrow{f} (Y, \psi)$

given by $X \xrightarrow{f} Y$ satisfying

$$\begin{tikzcd} SX \arrow[r, "Sf"] \arrow[d, "\phi"'] & SY \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

.

1. Kleisli for monads on categories (concretely)

$ob(A_S) = ob(A)$ $\quad S$

and ...

1. ... morphisms $X \nrightarrow Y$

- $f \in A(X, SY)$
- identities $X \xrightarrow{\eta_X} SX$
- composition $g \circ_S f :=$

$$X \xrightarrow{f} SY \xrightarrow{Sg} S^2 Z \xrightarrow{\mu_Z} SZ$$

2. ... morphisms $X \not\rightarrow Y$ given by morphisms of algebras

$$(SX, \mu_X) \longrightarrow (SY, \mu_Y)$$

3. ... morphisms generated by those in A + formal algebra structures $SX \xrightarrow{\varepsilon_X} X$ respected by all other morphisms.

Description 3 in more detail:

$$\begin{tikzcd} A \arrow[d, "S"'] \arrow[dr, "F_S"] & \\ A \arrow[r, "F_S"'] \arrow[ur, phantom, "\Uparrow\varepsilon"] & A_S \end{tikzcd}$$

such that $F_S \circ S \xrightarrow{\varepsilon} F_S$

is an algebra for

$-\circ S : [A, A_S] \longrightarrow [A, A_S]$

Generators

$$\frac{f \in A(X,Y)}{f : X \to Y} \quad \text{gen morphism}$$

$$\frac{X \in A}{SX \xrightarrow{\varepsilon_X} X} \quad \text{gen. morphism}$$

Relations

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, bend right, "1_X"'] & SX \arrow[d, "\varepsilon_X"] \\
 & X
\end{tikzcd}
\qquad
\begin{tikzcd}
S^2X \arrow[r, "\varepsilon_{SX}"] \arrow[d, "\mu_X"'] & SX \arrow[d, "\varepsilon_X"] \\
SX \arrow[r, "\varepsilon_X"'] & SX
\end{tikzcd}$$

$$\begin{tikzcd}
SX \arrow[r, "Sf"] \arrow[d, "\varepsilon_X"'] & SY \arrow[d, "\varepsilon_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The isomorphism of categories

$$\left\{\begin{array}{l}\text{gen.}\\ \text{+ relations}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{Kleisli maps}\\ \text{\& composition}\end{array}\right\}$$

$$X \xrightarrow{f} Y \quad \longmapsto \quad X \xrightarrow{f} Y \xrightarrow{\eta_Y} SY$$

$$SX \xrightarrow{\varepsilon_X} X \quad \longmapsto \quad SX \xrightarrow{1_{SX}} SX$$

$$\begin{tikzcd} X \arrow[r, "f"] & SY \arrow[r, "\varepsilon_Y"] & Y \\ & X \xrightarrow{f} SY \arrow[u, mapsto, red] & \end{tikzcd}$$

2. Kleisli for Pseudomonads

$$\underline{\text{Gray}} = \begin{cases} \text{2-categories} \\ \text{2-functors} \\ \text{pseudonat. transf.} \\ \text{modifications} \end{cases}$$

- A a $\underline{\text{2-category}}$

$$f 1_X \overset{=}{\cancel{\cong}} f$$

$$(hg)f \overset{=}{\cancel{\cong}} h(gf)$$

$$1_Y f \overset{=}{\cancel{\cong}} f$$

• $S : \mathcal{A} \longrightarrow \mathcal{A}$ a 2-functor

$$Sg\,Sf \overset{=}{\cancel{\cong}} S(gf)$$

$$1_{SX} \overset{=}{\cancel{\cong}} S1_X$$

• η, μ pseudonatural transformations

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\cong \eta_f"] & Y \arrow[d, "\eta_Y"] \\
SX \arrow[r, "Sf"'] & SY
\end{tikzcd}
\qquad
\begin{tikzcd}
S^2X \arrow[r, "Sf"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\cong \mu_f"] & S^2Y \arrow[d, "\mu_Y"] \\
SX \arrow[r, "Sf"'] & SY
\end{tikzcd}$$

$$
\begin{tikzcd}
S^3 \arrow[r, "S\mu"] \arrow[d, "\mu S"'] \arrow[dr, phantom, "\cong \alpha"] & S^2 \arrow[d, "\mu"] \\
S^2 \arrow[r, "\mu"'] & S
\end{tikzcd}
$$

$$
\begin{tikzcd}
S \arrow[r, "\eta S"] \arrow[dr, bend right, "1_S"'] \arrow[dr, phantom, "\cong \lambda"] & S^2 \arrow[d, "\mu"] \\
 & S
\end{tikzcd}
$$

$$
\begin{tikzcd}
S \arrow[r, "S\eta"] \arrow[dr, bend right, "1_S"'] \arrow[dr, phantom, "\cong \rho"] & S^2 \arrow[d, "\mu"] \\
 & S
\end{tikzcd}
$$

invertible modification

+ 2 axioms

(Marmolejo 1997)

- (Lack 1999) Eilenberg-Moore object for pseudo-monads described as a weighted limit

$\equiv$ { pseudoalgebras, pseudomorphisms, ps. algebra transformations

(Cheng, Hyland, Power 2003)

- Defined Kleisli bicategory

ie analogue of $X \xrightarrow{f} SY$ + Kleisli composition

- Showed $\mathrm{Bicat}(A_S, B)$

biequivalent to

$$\begin{tikzcd} A \arrow[d, "S"'] \arrow[dr] \arrow[d, phantom, shift left=4ex, "\Uparrow\varepsilon"] & \\ A \arrow[r] & B \end{tikzcd}$$

"pseudoalgebras"

This can't be a Gray-weighed colimit as it is a bicategory!

(Gambino, Lobbia 2021)

Observed that free pseudoalgebras are also not the enriched colimit as the comparison is only a biequivalence!

$$\mathrm{Gray}(A_s, B) \xrightarrow{\mathrm{Gray}(\varsigma B)} \mathrm{Gray}(A, B)$$

$$\mathcal{V} = (2\text{-Cat}, \otimes, 1)$$

$$\{\bullet \longrightarrow \bullet\} \otimes \left\{\begin{array}{c}\bullet\\ \downarrow\\ \bullet\end{array}\right\}$$

$$= \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

has internal hom

$$\mathrm{Gray}(A,B) = \left\{\begin{array}{l}\text{2-functors}\\ \text{ps. nat. transf.}\\ \text{modifications}\end{array}\right\}$$

& is locally presentable,

$\rightsquigarrow$ we can at least describe colimits via generators & relations

(as in 3rd description of Kleisli category.)

$$\mathrm{ob}(A_S) = \mathrm{ob}(A)$$

$$\frac{f \in A(X,Y)}{f : X \to Y \text{ generator}} \qquad \frac{X \in A_0}{SX \xrightarrow{\varepsilon_X} X \text{ generator}}$$

$$\frac{\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & Y \end{tikzcd} \in A}{\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & Y \end{tikzcd} \text{ gen 2-cell}}$$

\+ pseudoalgebra & pseudomorp

$X \in A$

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, bend right, "1_X"'] & SX \arrow[d, "\varepsilon_X"] \arrow[dl, phantom, "\cong u_X"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} S^2X \arrow[r, "\varepsilon_{SX}"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\cong m_X"] & SX \arrow[d, "\varepsilon_X"] \\ SX \arrow[r, "\varepsilon_X"'] & X \end{tikzcd}$$

gen. 2-cells.

$f \in A(X, Y)$

$$\begin{tikzcd} SX \arrow[r, "Sf"] \arrow[d, "\varepsilon_X"'] \arrow[dr, phantom, "\varepsilon_f \cong"] & SY \arrow[d, "\varepsilon_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

gen. 2-cell

hism relations.

Prop^n (M. 2023)

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, "F_S"'] \arrow[dr, "F"] & \\
\mathcal{A}_S \arrow[rr, "K"'] & & \mathcal{B}
\end{tikzcd}$$

$F \dashv U$ splitting $(\mathcal{A}, S)$

$\underbrace{K}$ is bi-fully faithful

ie $\mathcal{A}_S(X,Y) \longrightarrow \mathcal{B}(KX, KY)$ are equivalences of categories.

$\underline{\text{Th}^m}$ (M. 2023)

The following are equivalent

1. $F : A \to B$ is biess. surj on objects ie $\forall Y \in B$ $\exists X \in A$ $FX \simeq Y$

2. $K : A_S \to B$ is a biequivalence ie biess surj + bi fully faithful

3. For all 2-categories C

$$\mathrm{Gray}(B, C) \to \mathrm{Gray}(A_S, C) \to \mathrm{Gray}(A, C)^{\mathrm{Gray}(S, C)}$$

is a biequivalence.

$\iff$ biequivalence.

Example

$(\mathbb{1}, id_{\mathbb{1}})$ has one obj.

one generating morphism

$$\varepsilon : * \longrightarrow *$$

and two invertible gen.

2-cells $\quad 1_* \overset{\sim}{\underset{u}{\Rightarrow}} \varepsilon \overset{\sim}{\underset{m}{\Leftarrow}} \varepsilon^2$

satisfying monad laws.

Upshot

Enriched Kleisli left pseudoadjoints are not closed under composition!

$\rightsquigarrow$ Those satisfying the UP up to biequivalence $\Longleftrightarrow$ biess surj on objects which are closed under composition.

3. Nice homotopical Properties

Q: How do we make precise these weaker Universal Properties?

A: Tricategorical limits & Colimits

Q: Won't that be very complicated?

A: Yes, but we can think of these as homotopy limits & colimits enriched over $(\mathrm{Gray}, \otimes, 1)$ with Lack's model structure!

Lack's model structure

fibrations:
- lift adjoint eq.
- lift isos in hom-categories

w.e: biequivalences.

cofibrant obj: 2-categories whose underlying category is free on a graph.

gen. cofibrations:

$$\emptyset \xrightarrow{i_1} 1$$

$$\{\bullet \quad \bullet\} \xrightarrow{i_2} \{\bullet \longrightarrow \bullet\}$$

$$\left\{\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}\right\} \xrightarrow{i_3} \left\{\begin{tikzcd} \bullet \arrow[r, shift left=2, ""{name=U, below}] \arrow[r, shift right=2, ""{name=D, above}] \arrow[from=U, to=D, Rightarrow] & \bullet \end{tikzcd}\right\}$$

$$\left\{\begin{tikzcd} \bullet \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \bullet \end{tikzcd}\right\} \xrightarrow{i_4} \left\{\begin{tikzcd} \bullet \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}\right\}$$

<u>Thm</u> (M. 2023)

The weight for Kleisli objects is cofibrant in the projective model structure on

$$[\mathrm{Psmnd}, \mathrm{Gray}]$$

In fact, it can be built from the rep'ble using pushouts & copowers by $i_2, i_3, i_4 \ldots$

In particular,
$$\begin{tikzcd} \emptyset \arrow[d, "i_1"] \\ 1 \end{tikzcd}$$
& retracts are not needed.

Construction of EM

$$\begin{tikzcd}
 & A \arrow[dr, "s"] \arrow[d, phantom, "\Downarrow \phi"] & \\
\mathrm{Ins}_1 \arrow[ur, "p"] \arrow[r, "p"'] & A \arrow[r, "1_A"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Ins}_1 \arrow[r, "\phi"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^2 \arrow[d] \\
A \arrow[r, "{(1_A, s)}"'] & A \times A
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{Ins} \arrow[dr, "p"] & \\
\mathrm{Ins}_2 \arrow[ur, bend left] \arrow[dr, bend right] \arrow[r, phantom, "\phi \circ \eta \Downarrow \quad \equiv \quad \Downarrow 1"] & & A \\
 & \mathrm{Ins} \arrow[ur, "p"'] &
\end{tikzcd}$$

Dual construction of weight

$$
\begin{tikzcd}
\mathrm{Rep} + \mathrm{Rep} \cong (1+1) \odot \mathrm{Rep} \arrow[r, "i_2 \odot \mathrm{Rep}"] \arrow[d, "(1,s)"'] & \{\bullet \to \bullet\} \odot \mathrm{Rep} \arrow[d] \\
\mathrm{Rep} \arrow[r] & \mathrm{Ins}_1 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\{\bullet \rightrightarrows \bullet\} \odot \mathrm{Ins}_1 \arrow[r, "i_3 \odot \mathrm{Ins}_1"] \arrow[d] & \{\bullet \overset{\to}{\underset{\to}{\perp}} \bullet\} \odot \mathrm{Ins}_1 \arrow[d] \\
\mathrm{Ins}_1 \arrow[r] & \mathrm{Ins}_2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

4. Tricategorical Universal Properties

$$A \xrightarrow{F} B$$

$$A^{op} \xrightarrow{W} \mathrm{Bicat}$$

Def^n (Power 2007)

$$\mathrm{Tricat}(A^{op}, \mathrm{Bicat})(W, B(F-, X))$$

$$\sim \quad B(W \circ F, X)$$

In (Campbell 2016) & (M. 2023) these are related to "nice" * Gray enriched weighted colimits

* $A(X,Y)$ are cofibrant,

*
$$
\begin{tikzcd}
A^{op} \arrow[rr, "W"] \arrow[dr, dashed, "W'"'] & & \mathrm{Bicat} \\
& \mathrm{Gray} \arrow[ur, "inc"'] &
\end{tikzcd}
$$

projective cofibrant

Th^m (M. 2023) Gray-enriched Kleisli object of a pseudomonad is also its tricategorical colimit weighted by

$$\mathrm{Psmnd}^{op} \longrightarrow \mathrm{Gray}$$

$$(\overline{\Delta}_+, -\oplus 1)$$

<u>Cor</u> (M. 2023) The Kleisli bicategory & 2-category of free pseudoalgebras are both also "trikleisli" objects of the pseudomonad.

5. Applications / Further Work

- Free cocompletions for trikleisli objects / FTPsM2.

- Symmetric / Sylleptic / Braided monoidal structures on pseudoalgebras / Kleisli bicategories.

Status

- Explicit description given. UP of cocompletion open.
- Applied to double-cats, (M. 2023) & to Categorical systems theory (Capucci, Myers 2024)

- Liftings & Extensions shown in arxiv 2402.11703 & in an upcoming preprint.
- Monoidal Kleisli treated by (Saville, Pacquet 2024)
- tricategorical UP established.