Representable Behaviour in Double Categorical Systems Theory

Old and new wisdom

Matteo Capucci
University of Strathclyde / ARIA Creator

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Double Categorical Systems Theory

DCST (Myers 2020; Myers 2021) is a principled mathematical framework for the ontology and phenomenology of systems, and distills lots of wisdom from various other categorical approaches. To name a few:

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3. **Bicategories of transition systems** (Katis, Sabadini, and Walters 1997a; Katis, Sabadini, and Walters 1997b; Katis, Sabadini, and Walters 2002; Gianola, Kasangian, and Sabadini 2017; Di Lavore, Gianola, Román, Sabadini, and Sobociński 2021)

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4. **Double categories of structured cospans** (Fiadeiro and Schmitt 2007; Fong 2015; Baez and Courser 2020; Baez, Courser, and Vasilakopoulou 2022; Baez and Master 2020)

Double Categorical Systems Theory

In DCST, **systems** are organized as algebras of a symmetric double operad, or symmetric monoidal double category of **composition operations** or **processes**.

$$
1 \overset{\mathbf{Sys}}{\nrightarrow} \mathbb{I} \qquad\qquad
\begin{array}{c} \mathsf{S} \\ \downarrow{\scriptstyle \varphi} \\ \mathsf{S}' \end{array}
\;\bullet\;
\begin{array}{ccc} I & \overset{p}{\longrightarrow\!\!\!\!\!\bullet\;} & J \\ {\scriptstyle h}\downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle k} \\ I' & \underset{p'}{\longrightarrow\!\!\!\!\!\bullet\;} & J' \end{array}
\;=\;
\begin{array}{c} \mathsf{S} \bullet p \\ \downarrow{\scriptstyle \varphi \bullet \alpha} \\ \mathsf{S}' \bullet p' \end{array}
\tag{0.1}
$$

This is an etymologically accurate structure (*system* meaning ‘composed of things’).

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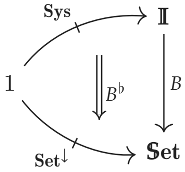
This is an etymologically accurate structure (*system* meaning ‘composed of things’).
Behaviours are then functors out of them:

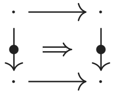


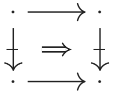
Plan of the talk

1. Theories of composition and theories of systems
   1.1 Composition theories as symmetric monoidal double categories
   1.2 Systems theories as right modules
   1.3 Examples: theories from adequate triples, Moore machines as free theories
2. Representable behaviour
   2.1 Functorial behaviour
   2.2 Compositionality theorem in behavioural form
   2.3 Multi- and plurirepresentable behaviour, nerve behaviour

Some conventions

1. Double categories are **weak** by default, (double) functors are **lax** by default
2. For the rest I mostly follow
   - M. Grandis, *Higher Dimensional Categories: From Double to Multiple Categories*. World Scientific, 2019
3. '(Loose) arrows' are marked ($\multimap\!\!\rightarrow$ or $\nrightarrow$), '(tight) morphisms' are not ($\longrightarrow$):



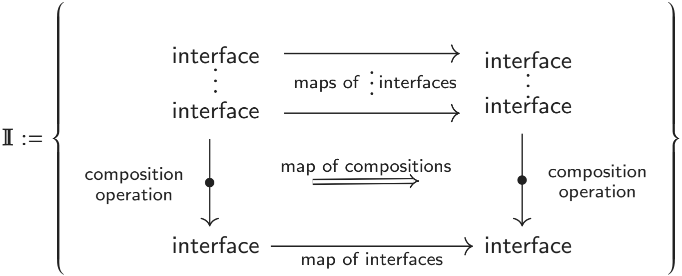


Theories of composition
& theories of systems

Theories of compositions

Definition

A **theory of composition** $\mathbb{I}$ is an isofibrant *symmetric double operad* with the attitude described below:



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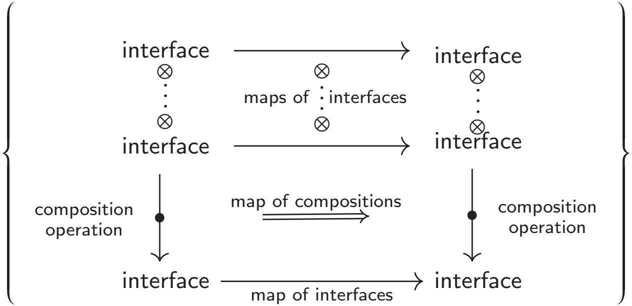

We will assume our theories **representable**, hence **symmetric monoidal double categories**.

Theories of systems

Definition

A **theory of systems** over the theory of composition $\mathbb{I}$ is

(tight datum) a **displayed symmetric monoidal category**, i.e. a strict monoidal isofibration:

$$\begin{array}{ccc} \mathbf{Sys} & \qquad & \mathsf{S} \xrightarrow{\varphi} \mathsf{S}' \\ {\scriptstyle D}\big\downarrow & & \\ \mathbb{I}_0 & & I \xrightarrow{h} I' \end{array}$$

and we write $\mathsf{S} \in \mathbf{Sys}(I)$, $\varphi \in \mathbf{Sys}(h)$.

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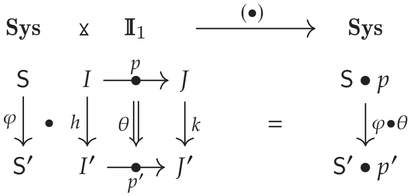
(module structure) equipped with a (right) **module structure**, i.e. a strong monoidal functor:

$$\begin{array}{ccccc} \mathbf{Sys} & \longleftarrow & \mathbf{Sys} \times \mathbb{I}_1 & \overset{(\bullet)}{\dashrightarrow} & \mathbf{Sys} \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ \mathbb{I}_0 & \xleftarrow{s} & \mathbb{I}_1 & \xrightarrow{t} & \mathbb{I}_0 \end{array}$$

Theories of systems

The module structure amounts to an operation



with coherent structure morphisms

unitor $\mathsf{S} \bullet 1 \cong \mathsf{S}$,

compositor $(\mathsf{S} \bullet p) \bullet q \cong \mathsf{S} \bullet (p \odot q)$,

interchangers $(\mathsf{S} \bullet p) \otimes (\mathsf{R} \bullet q) \cong (\mathsf{S} \otimes \mathsf{R}) \bullet (p \otimes q)$

Example: behavioural theories

Example

For any *finitely complete category* $\mathbf{E}$, $\mathbb{S}\mathbf{pan}(\mathbf{E})$ is a theory of composition and $\mathbf{E}^{\downarrow} \xrightarrow{\partial_1} \mathbf{E}$ supports a $\mathbb{S}\mathbf{pan}(\mathbf{E})$-module structure, given by pull-push (denoted $\bowtie$)

$$
\begin{array}{ccc}
S & \xrightarrow{f} & A \\
{\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle h} \\
S' & \xrightarrow[f']{} & A'
\end{array}
\quad \bowtie \quad
\begin{array}{ccccc}
A & \xleftarrow{l} & P & \xrightarrow{r} & B \\
{\scriptstyle h}\downarrow & & {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle k} \\
A' & \xleftarrow[l']{} & P' & \xrightarrow[r']{} & B'
\end{array}
\quad = \quad
\begin{array}{ccc}
S \bowtie P & \xrightarrow{f \bowtie (l,r)} & B \\
{\scriptstyle\varphi\bowtie\theta}\downarrow & & \downarrow{\scriptstyle k} \\
S' \bowtie P' & \xrightarrow[f' \bowtie (l',r')]{} & B'
\end{array}
$$

This is the **behavioural theory** associated to $\mathbf{E}$. We keep denoting it as $\mathbf{E}$, chiefly in the case $\mathbf{E} = \mathbf{Set}$.

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$$\begin{array}{ccc} S & \xrightarrow{f} & A \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle h} \\ S' & \xrightarrow[f']{} & A' \end{array} \quad \ltimes \quad \begin{array}{ccccc} A & \xleftarrow{l} & P & \xrightarrow{r} & B \\ {\scriptstyle h}\downarrow & & {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle k} \\ A' & \xleftarrow[l']{} & P' & \xrightarrow[r']{} & B' \end{array} \quad = \quad \begin{array}{ccc} S \ltimes P & \xrightarrow{f \ltimes (l,r)} & B \\ {\scriptstyle \varphi\ltimes\theta}\downarrow & & \downarrow{\scriptstyle k} \\ S' \ltimes P' & \xrightarrow[f' \ltimes (l',r')]{} & B' \end{array}$$

This is the **behavioural theory** associated to $\mathbf{E}$. We keep denoting it as $\mathbf{E}$, chiefly in the case $\mathbf{E} = \mathbb{S}\mathbf{et}$.

Similarly, if $\mathbf{E}$ is *regular* $\mathbf{E}^{\downarrowtail} \xrightarrow{\partial_1} \mathbf{E}$ is a right module over $\mathbb{R}\mathbf{el}(\mathbf{E})$. We call it the **blackbox behavioural theory** associated to $\mathbf{E}$.

Example: adequate triples

Definition (following Haugseng, Hebestreit, Linskens, and Nuiten 2023)

A **symmetric monoidal adequate triple** is a symmetric monoidal category $(\mathbf{E}, \otimes)$ equipped with two wide subcategories[1] whose morphisms are called *ingressive* $\rightarrowtail$ and *egressive* $\twoheadrightarrow$, such that:

1. every isomorphism is ingressive,
2. ingressive and egressive maps are closed under monoidal products,
3. every cospan as below left can be completed to a pullback as below right:

$$\begin{array}{ccc} & & \cdot \\ & & \downarrow\!\!\downarrow \\ \cdot & \rightarrowtail & \cdot \end{array} \quad \mapsto \quad \begin{array}{ccc} \cdot & \rightarrowtail & \cdot \\ \downarrow\!\!\downarrow & \lrcorner & \downarrow\!\!\downarrow \\ \cdot & \rightarrowtail & \cdot \end{array} \tag{0.2}$$

4. and $\otimes$ commutes with ingressive-egressive pullbacks,

i.e.: $E^{\twoheaddownarrow} \xrightarrow{\partial_1} E$ is a strict monoidal isofibration admitting strong cartesian lifts of every ingressive map.

Example: adequate triples

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For every symmetric monoidal adequate triple $(\mathbf{E}, \rightarrowtail, \twoheadrightarrow)$, $\mathbb{S}\mathbf{pan}(\mathbf{E})$ is a theory of composition and $\mathbf{E}^{\downarrow} \xrightarrow{\partial_1} \mathbf{E}$ supports a $\mathbb{S}\mathbf{pan}(\mathbf{E}, \rightarrowtail, \twoheadrightarrow)$-module structure:

$$
\begin{array}{ccc} S & \xrightarrow{f} & A \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle h} \\ S' & \xrightarrow[f']{} & A' \end{array}
\quad \mathbin{⧖} \quad
\begin{array}{ccccc} A & \xleftarrow{l} & P & \xrightarrow{r} & B \\ {\scriptstyle h}\downarrow & & {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle k} \\ A' & \xleftarrow[l']{} & P' & \xrightarrow[r']{} & B' \end{array}
\quad = \quad
\begin{array}{ccc} S \mathbin{⧖} P & \xrightarrow{f \mathbin{⧖} (l, r)} & B \\ {\scriptstyle\varphi \mathbin{⧖} \theta}\downarrow & & \downarrow{\scriptstyle k} \\ S' \mathbin{⧖} P' & \xrightarrow[f' \mathbin{⧖} (l', r')]{} & B' \end{array}
$$

Example: P-charts & P-lenses

Let $P : \mathbf{E} \longrightarrow \mathbf{B}$ be a strict symmetric monoidal fibration. We represent $\mathbf{E}$ as P-**charts**:

$$\begin{pmatrix} A^- \\ A^+ \end{pmatrix} \overset{h^\flat}{\underset{h}{\rightrightarrows}} \begin{pmatrix} A'^- \\ A'^+ \end{pmatrix} = \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \xRightarrow{\begin{pmatrix} h^\flat \\ A^+ \end{pmatrix}} \begin{pmatrix} A'^- \\ A^+ \end{pmatrix} \xRightarrow{\begin{pmatrix} A'^- \\ h \end{pmatrix}} \begin{pmatrix} A'^- \\ A'^+ \end{pmatrix}$$

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Turns out $(\mathbf{E}, \mathrm{vert}, \mathrm{cart})$ is a symmetric monoidal adequate triple, thus we can define:

$$\mathbb{S}\mathbf{pan}(P) := \mathbb{S}\mathbf{pan}(\mathbf{E}, \mathrm{vert}, \mathrm{cart}) = \left\{ \begin{array}{ccc} \begin{pmatrix}A^-\\A^+\end{pmatrix} & \xRightarrow{\begin{pmatrix}h^\flat\\h\end{pmatrix}} & \begin{pmatrix}A'^-\\A'^+\end{pmatrix} \\ \begin{pmatrix}f^\sharp\\A^+\end{pmatrix} \Uparrow & & \Uparrow \begin{pmatrix}f'^\sharp\\A'^+\end{pmatrix} \\ \begin{pmatrix}B^-\\A^+\end{pmatrix} & \xRightarrow{\begin{pmatrix}\theta^\flat\\\theta\end{pmatrix}} & \begin{pmatrix}B'^-\\A'^+\end{pmatrix} \\ \begin{pmatrix}B^-\\f\end{pmatrix} \Downarrow & & \Downarrow \begin{pmatrix}B'^-\\f'\end{pmatrix} \\ \begin{pmatrix}B^-\\B^+\end{pmatrix} & \xRightarrow{\begin{pmatrix}k^\flat\\k\end{pmatrix}} & \begin{pmatrix}B'^-\\B'^+\end{pmatrix} \end{array} \right\}$$

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always a thin double category!

Example: P-charts & P-lenses

We denote a span as above with the shorter notation:

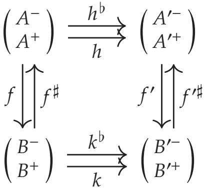
$$\begin{pmatrix} A^- \\ A^+ \end{pmatrix} \overset{f^\sharp}{\underset{f}{\leftrightarrows}} \begin{pmatrix} B^- \\ B^+ \end{pmatrix} \; := \; \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \xLeftarrow{\begin{pmatrix} f^\sharp \\ A^+ \end{pmatrix}} \begin{pmatrix} B^- \\ A^+ \end{pmatrix} \xRightarrow{\begin{pmatrix} B^- \\ f \end{pmatrix}} \begin{pmatrix} B^- \\ B^+ \end{pmatrix}$$

This is a P-**lens** (Spivak 2022; Capucci, Gavranović, Malik, Rios, and Weinberger 2024).

The category of P-lenses associated to $\mathbf{Set}^{\downarrow} \xrightarrow{\partial_1} \mathbf{Set}$ is equivalent to $\mathbf{Poly}$ (Niu and Spivak 2023).

Example: P-charts & P-lenses

Therefore, P-charts and P-lenses form a thin double category $\mathbb{L}\mathbf{ens}(P) \equiv \mathbb{S}\mathbf{pan}(P)$, whose squares are as above and denoted as below:



In type-theoretic notation, these encode the following commutativity condition:

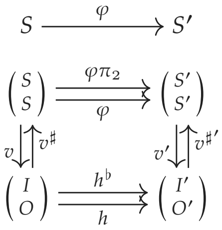
$$\begin{aligned} \forall a^+ : A^+, \qquad & k(f(a^+)) = f'(h(a^+)), \\ a^+ : A^+ \vdash \ \forall b^- : B^-(f(a^+)), \qquad & h^\flat(a^+, f^\sharp(a^+, b^-)) = f'^\sharp(h(a^+), k^\flat(f(a^+), b^-)). \end{aligned} \tag{0.3}$$

Example: Moore machines

On $\mathbb{L}\mathbf{ens}(\mathbf{Set}^{\downarrow} \xrightarrow{\partial_1} \mathbf{Set})$ we consider two different theories of systems:

1. **deterministic discrete Moore machines** $\mathbf{Moore}(\mathbf{Set})$, where a Moore machine over $\binom{I}{O}$ is a lens as below left and a morphism of Moore machines is a map φ (over the chart $\binom{h^\flat}{h}$) making the square below commute:

$$\binom{S}{S} \overset{v^\sharp}{\underset{v}{\leftrightarrows}} \binom{I}{O} \equiv \begin{cases} v : S \to O, \\ v^\sharp : (s : S) \times I(v(s)) \to S \end{cases}$$



The module structure is given by composition of lenses and (looseward) composition of squares.

Example: Moore machines

On $\mathbb{L}\mathbf{ens}(\mathbf{Set}^{\downarrow} \xrightarrow{\partial_1} \mathbf{Set})$ we consider two different theories of systems:

1. **deterministic discrete Moore machines $\mathbf{Moore}(\mathbf{Set})$**
2. **possibilistic discrete Moore machines $\mathbf{Moore}_{\mathcal{P}}(\mathbf{Set})$** are similarly defined, except now a Moore machine is given by a non-deterministic lens; while a map is square which commutes only up to containment:

$$\begin{pmatrix} S \\ S \end{pmatrix} \underset{v}{\overset{v^\sharp}{\leftrightarrows}} \begin{pmatrix} I \\ O \end{pmatrix} \equiv \begin{cases} v : S \to O, \\ v^\sharp : (s : S) \times I(v(s)) \to \mathcal{P} S \end{cases}$$

$$S \xrightarrow{\varphi} S'$$

$$\begin{array}{ccc} \begin{pmatrix} S \\ S \end{pmatrix} & \underset{\varphi}{\overset{\varphi\pi_2}{\rightrightarrows}} & \begin{pmatrix} S' \\ S' \end{pmatrix} \\ v \downarrow\uparrow v^\sharp & \subseteq & v' \downarrow\uparrow v^{\sharp\prime} \\ \begin{pmatrix} I \\ O \end{pmatrix} & \underset{h}{\overset{h^\flat}{\rightrightarrows}} & \begin{pmatrix} I' \\ O' \end{pmatrix} \end{array}$$

Intuitively: the transitions out of $\varphi(s) \in S'$ must contain *at least* the image of those out of $s \in S$.

Example: free theories

Given a displayed symmetric monoidal category $T : \mathbf{X} \to \mathbb{I}_0$, the **free theory on** T is $T \mathbin{\not\bullet} \mathbb{I} := \mathbb{I}[T] := T \rtimes \mathbb{I}$:

$$
\begin{array}{ccccc}
\mathbf{X} \rtimes \mathbb{I}_1 & \longleftarrow & \mathbf{X} \rtimes \mathbb{I}_1 \rtimes \mathbb{I}_1 & \overset{\mathbf{X} \rtimes (\odot)}{\dashrightarrow} & \mathbf{X} \rtimes \mathbb{I}_1 \\
\big\downarrow{\scriptstyle T \rtimes \mathbb{I}} & \lrcorner & \big\downarrow{\scriptstyle T \rtimes \mathbb{I}} & & \big\downarrow{\scriptstyle T \rtimes \mathbb{I}} \\
\mathbb{I}_0 & \overset{s}{\longleftarrow} & \mathbb{I}_1 & \overset{t}{\longrightarrow} & \mathbb{I}_0
\end{array}
$$

Systems over $J \in \mathbb{I}$ are given by 'formal composites' of generators $\mathsf{G} \in \mathbf{X}(I)$ and a process $I \overset{p}{\longrightarrow\!\!\bullet\!\!\rightarrow} J$.

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\downarrow{\scriptstyle T\times\mathbb{I}} & \lrcorner & \downarrow{\scriptstyle T\times\mathbb{I}} & & \downarrow{\scriptstyle T\times\mathbb{I}} \\
\mathbb{I}_0 & \xleftarrow{\quad s \quad} & \mathbb{I}_1 & \xrightarrow{\quad t \quad} & \mathbb{I}_0
\end{array}$$

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Example

Given a section $T : \mathbf{B} \to \mathbf{E}$ of a fibration $P : \mathbf{E} \to \mathbf{B}$, the free theory $T \mathbin{\rlap{/}\bullet} \mathbb{L}\mathbf{ens}(P)$ is the **theory of (generalized) Moore machines** (this construction is central in (Myers 2021)):

$$\underbrace{\begin{pmatrix} TS \\ S \end{pmatrix}}_{\text{generator}} \overset{v^\sharp}{\underset{v}{\leftrightarrows}} \begin{pmatrix} I \\ O \end{pmatrix}.$$

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\downarrow{\scriptstyle T \dot\times \mathbb{I}} & \lrcorner & \downarrow{\scriptstyle T \dot\times \mathbb{I}} & & \downarrow{\scriptstyle T \dot\times \mathbb{I}} \\
\mathbb{I}_0 & \xleftarrow{\;s\;} & \mathbb{I}_1 & \xrightarrow{\;t\;} & \mathbb{I}_0
\end{array}
$$

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$$
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$$

Notable instances are: open ODEs being free on T a tangent structure on $\mathbf{B}$, $\mathbf{Moore}(\mathbf{Set})$ being free on $S \overset{T}{\mapsto} S \times S \xrightarrow{\pi_1} S$.

Example: free theories

Given a displayed symmetric monoidal category $T : \mathbf{X} \to \mathbb{I}_0$, the **free theory on** T is $T \mathbin{\not\bullet} \mathbb{I} := \mathbb{I}[T] := T \ltimes \mathbb{I}$:

$$\begin{array}{ccccc}
\mathbf{X} \ltimes \mathbb{I}_1 & \longleftarrow & \mathbf{X} \ltimes \mathbb{I}_1 \ltimes \mathbb{I}_1 & \overset{\mathbf{X} \ltimes (\odot)}{\dashrightarrow} & \mathbf{X} \ltimes \mathbb{I}_1 \\
\downarrow {\scriptstyle T \ltimes \mathbb{I}} & \lrcorner & \downarrow {\scriptstyle T \ltimes \mathbb{I}} & & \downarrow {\scriptstyle T \ltimes \mathbb{I}} \\
\mathbb{I}_0 & \xleftarrow{\quad s \quad} & \mathbb{I}_1 & \xrightarrow{\quad t \quad} & \mathbb{I}_0
\end{array}$$

Systems over $J \in \mathbb{I}$ are given by ‘formal composites’ of generators $\mathsf{G} \in \mathbf{X}(I)$ and a process $I \overset{p}{\twoheadrightarrow} J$.

Example

Given a section $T : \mathbf{B} \to \mathbf{E}$ of a fibration $P : \mathbf{E} \to \mathbf{B}$, the free theory $T \mathbin{\not\bullet} \mathbb{L}\mathrm{ens}(P)$ is the **theory of (generalized) Moore machines** (this construction is central in (Myers 2021)):

$$\underbrace{\begin{pmatrix} TS \\ S \end{pmatrix}}_{\text{generator}} \underset{v}{\overset{v^\sharp}{\leftrightarrows}} \begin{pmatrix} I \\ O \end{pmatrix}.$$

Notable instances are: open ODEs being free on T a tangent structure on $\mathbf{B}$, $\mathbf{Moore}(\mathbf{Set})$ being free on $S \overset{T}{\mapsto} S \times S \xrightarrow{\pi_1} S$. **Beware!** $\mathbf{Moore}_{\mathcal{P}}(\mathbf{Set})$ is not free but it's subfree.

Functorial behaviour

Functorial behaviour

Idea: while *theories of systems* describe the structural (morphological & compositional) aspects of systems, *functors* out of them describe their behavioural/dynamical aspects:

$$B : \mathbf{Sys} \longrightarrow \mathbf{E}$$

Usually, the codomain is a (you guessed it) behavioural theory.

This is a form of **functorial semantics**, since the functor itself establishes a relationship between two theories in which the domain is 'interpreted' in the codomain.

Morphisms of systems theories

Definition

A **lax morphism of systems theories** $\begin{pmatrix} F^\flat \\ F \end{pmatrix} : \begin{pmatrix} \mathbf{Sys} \\ \mathbb{I} \end{pmatrix} \rightrightarrows \begin{pmatrix} \mathbf{Sys}' \\ \mathbb{I}' \end{pmatrix}$ is given by

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$$\mathbb{I} \xrightarrow{F} \mathbb{I}' \qquad \begin{array}{ccc} \mathbf{Sys} & \xrightarrow{F^\flat} & \mathbf{Sys}' \\ \downarrow & & \downarrow \\ \mathbb{I}_0 & \xrightarrow{F_0} & \mathbb{I}'_0 \end{array}$$

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(laxators) and suitably coherent laxators as below:

monoidal laxators $1' \xrightarrow{\upsilon} F1, \quad FI \otimes' FJ \xrightarrow{\nu} F(I \otimes J)$

$1' \xrightarrow{\upsilon^\flat} F^\flat 1, \quad F^\flat(\mathsf{S}) \otimes' F^\flat(\mathsf{R}) \xrightarrow{\nu^\flat} F^\flat(\mathsf{S} \otimes \mathsf{R})$

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$\begin{pmatrix} F^\flat \\ F \end{pmatrix}$ is **strong monoidal/compositional** when the corresponding laxators are invertible.

Theory of behaviour

Definition

A **theory of behaviour** $\binom{B^\flat}{B} : \binom{\mathbf{Sys}}{\mathbb{I}} \longrightarrow \binom{\mathbf{Set}^\downarrow}{\mathbb{S}\mathbf{et}}$ is given by

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Aside: obstructions to compositionality

One can classify obstructions to monoidality/compositionality by factoring the laxators, e.g. for ℓ:

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Very general idea! Works for any finitely complete category $\mathbf{E}$ equipped with a *modality* $\blacksquare$, e.g. a lex reflective subcategory.

Representable behaviour

Representability allows to tame the complexity of a theory of behaviour & it is very common in nature.

Definition

A **representable theory of behaviour** over $\mathbf{Sys}$ is one given by

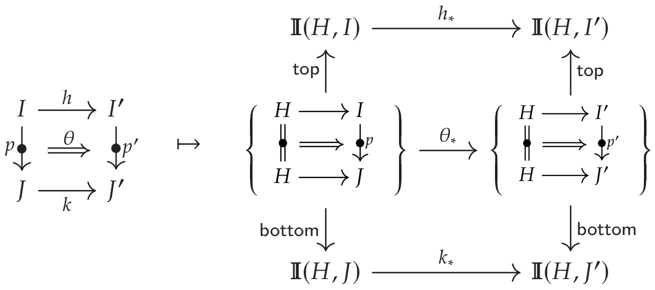
$$\begin{pmatrix} \mathbf{Sys}(\mathsf{C}, -) \\ \mathbb{I}(H, -) \end{pmatrix}$$

for some commutative comonoidal system $\mathsf{C} \in \mathbf{Sys}(H)$.

We think of C as a **clock**, with interface H being its '**hands**'.

Representable behaviour

On interfaces, $\mathbf{Sys}(\mathsf{C},-)$ is given by the Parè representable at $\mathbb{I}(H,-) : \mathbb{I} \to \mathbb{S}\mathbf{et}$:



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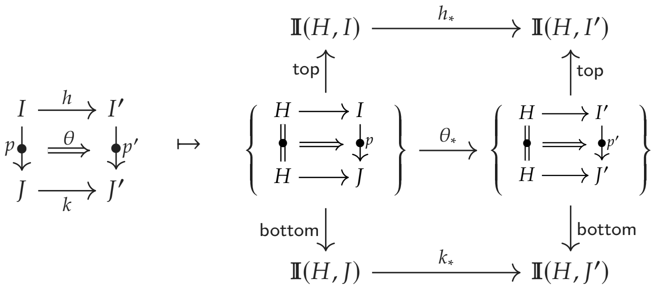

The comonoid structure (ε, Δ) on H defines the monoidal laxators:

$$1 \xrightarrow{\mathrm{id}_1} \mathbb{I}(\mathbf{1}, \mathbf{1}) \xrightarrow{\varepsilon^*} \mathbb{I}(H, \mathbf{1}), \quad \mathbb{I}(H, I) \times \mathbb{I}(H, J) \xrightarrow{(\otimes)} \mathbb{I}(H \otimes H, I \otimes J) \xrightarrow{\Delta^*} \mathbb{I}(H, I \otimes J)$$

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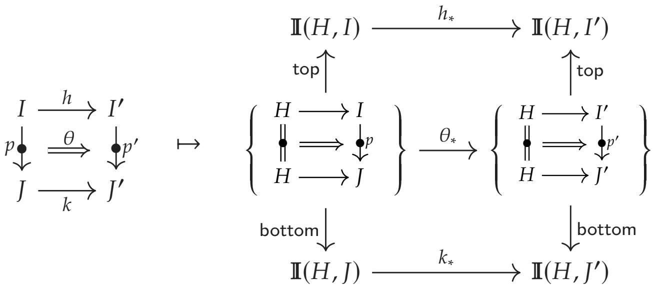

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The compositional laxators are induced by looseward identity/composition of squares.

Representable behaviour

Similarly, on systems, we get a functor $\mathbf{Sys}(\mathsf{C}, -) : \mathbf{Sys} \to \mathbf{Set}/\mathbb{I}(H, -)$.

$$\mathsf{S} \xrightarrow{\varphi} \mathsf{S}' \mapsto \begin{array}{ccc} \mathbf{Sys}(\mathsf{C},\mathsf{S}) & \xrightarrow{\varphi_*} & \mathbf{Sys}(\mathsf{C},\mathsf{S}') \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ \mathbb{I}(H, I) & \xrightarrow[(D\varphi)_*]{} & \mathbb{I}(H, I') \end{array}$$

Again, the comonoid structure of C induces monoidal laxators, and the compositional laxators are given by composition:

$$\mathbf{Sys}(\mathsf{C},\mathsf{S}) \times \mathbb{I}(H, p) \xrightarrow{\ell} \mathbf{Sys}(\mathsf{C},\mathsf{S} \bullet p)$$

$$\begin{array}{ccccc} \mathsf{C} & & H & = & H \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle h}\downarrow & \theta\Downarrow & \downarrow{\scriptstyle k} \\ \mathsf{S} & & I & \xrightarrow[p]{} & J \end{array} \longmapsto \begin{array}{c} \mathsf{C} \\ \downarrow{\scriptstyle\eta} \\ \mathsf{C} \bullet 1_H \\ \downarrow{\scriptstyle\varphi\bullet\theta} \\ \mathsf{S} \bullet p \end{array}$$

Representable behaviour for non-/deterministic Moore machines

Example

The theory of trajectories is representable by the walking trajectory

$$\mathsf{T}_\omega := \ 0 \rightsquigarrow 1 \rightsquigarrow 2 \rightsquigarrow \cdots$$

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The theory of fixpoints is representable by the walking fixpoint $\mathsf{L}_1 := \; 0 \circlearrowleft$.

Similarly, the theory of n-th order cycles are represented by walking loops $\mathsf{L}_n \in \mathbf{Moore}_{\mathcal{P}}\binom{1}{n}$.

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Non-example

The theory of states $\mathbf{Moore}_{\mathcal{P}}\binom{I}{O} \ni \mathsf{S} \xrightarrow{\varphi} \mathsf{S}' \mapsto S \xrightarrow{\varphi} S \in \mathbf{Set}/1$ is represented by the initial system $0 \in \mathbf{Moore}_{\mathcal{P}}\binom{\varnothing}{\varnothing}$.

States *with observations* are represented by $\mathsf{L}_0 := \; 0$.

Compositionality of representable behaviours

Compositionality of representable behaviours hinges on three properties:

1. $\mathbb{I}$ and $\mathbf{Sys}$ are **cartesian**, in which case

$$\begin{pmatrix}\mathsf{C} \\ H\end{pmatrix} \overset{\exists!}{\dashrightarrow} \begin{pmatrix}1 \\ \mathbf{1}\end{pmatrix} \qquad \begin{pmatrix}\mathsf{C} \\ H\end{pmatrix} \to \begin{pmatrix}\mathsf{S} \\ I\end{pmatrix} \times \begin{pmatrix}\mathsf{R} \\ J\end{pmatrix} = \left(\begin{pmatrix}\mathsf{C} \\ H\end{pmatrix} \overset{\exists!}{\dashrightarrow} \begin{pmatrix}\mathsf{S} \\ I\end{pmatrix}, \begin{pmatrix}\mathsf{C} \\ H\end{pmatrix} \overset{\exists!}{\dashrightarrow} \begin{pmatrix}\mathsf{R} \\ J\end{pmatrix} \right)$$

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   This is more common than it looks: **all the examples we mentioned so far are cartesian**.

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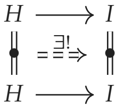

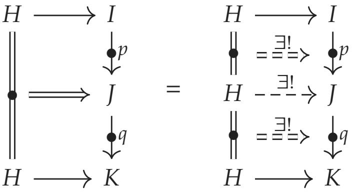

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2. $\mathbb{I}$ is **spanlike**, in which case **the compositional laxators are invertible**:

$$1 \xrightarrow{\sim} \mathbb{I}(H, 1), \qquad \mathbb{I}(H, p) \times \mathbb{I}(H, q) \xrightarrow{\sim} \mathbb{I}(H, p \odot q)$$

   This too is the case for **all the composition theories we mentioned so far** (because they are literally double categories of spans).

Compositionality of representable behaviours

3. **Sys** is **observable**, in which case

$$
\begin{array}{c}
\mathsf{C} \xrightarrow{\varphi} \mathsf{S} \bullet p \\[2em]
H \xrightarrow[k]{} J
\end{array}
\quad = \quad
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This is rarely the case!

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This is rarely the case!

All the above properties need not hold for the entirety of $\mathbf{Sys}$ and $\mathbb{I}$, it's enough they hold 'at $\binom{\mathsf{C}}{H}$':

Definition

We say $\binom{\mathsf{C}}{H}$ **is cartesian/spanlike/observable** when the corresponding laxators for the representable $\binom{\mathsf{C}}{H}$ are invertible.

Observability of a system in $\mathbf{E}^{\downarrow}$

Let $\mathbf{E}^{\downarrow}$ be the theory associated to an adequate triple.It is cartesian and spanlike, but in general not observable.

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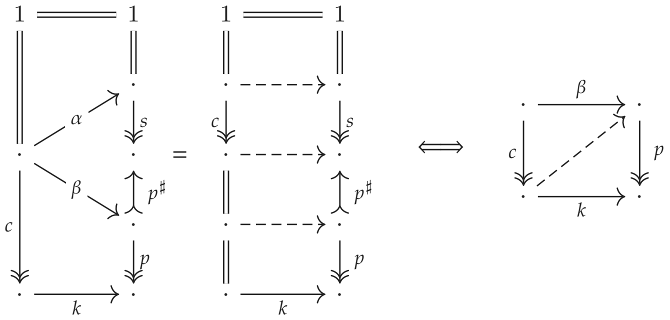

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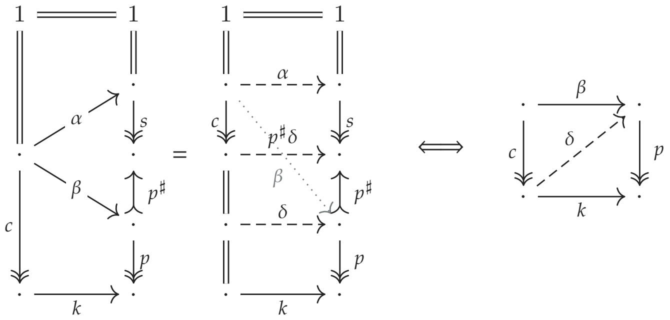

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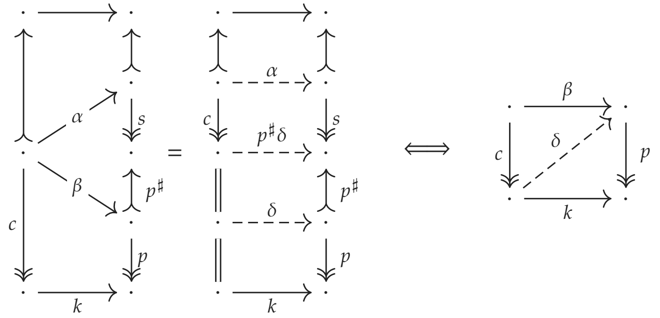
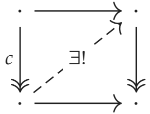


Observability of a system in $\mathbf{E}^{\twoheaddownarrow}$

Theorem

Let $\mathbf{E}$ be a symmetric monoidal adequate triple, $T : \mathbf{X} \to \mathbf{E}$ displayed symmetric monoidal category.

A system $TC \overset{c^\sharp}{\leftarrowtail} \cdot \overset{c}{\twoheadrightarrow} H$ is observable in the free theory $T \oint \mathbb{S}\mathbf{pan}(\mathbf{E})$ iff c is left orthogonal to all egressive maps:



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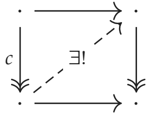

Corollary ($\Leftarrow$ is Myers 2021, Theorem 5.3.3.1)

For (P, T) *theory of Moore machines, recall* $T \not\bullet \mathbb{S}\mathbf{pan}(P) = \mathbf{Moore}(P, T)$, *thus* $\begin{pmatrix} TC \\ C \end{pmatrix} \underset{c}{\overset{c^\sharp}{\leftrightarrows}} \begin{pmatrix} H^- \\ H^+ \end{pmatrix}$ *is observable iff* c *is invertible.*

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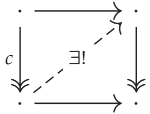

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Corollary

If c is split epi, $\begin{pmatrix} TC \\ C \end{pmatrix} \underset{c}{\overset{c^\sharp}{\leftrightarrows}} \begin{pmatrix} H^- \\ H^+ \end{pmatrix}$ induces surjective laxators, i.e. there are no 0-generative effects.

Multirepresentable behaviour

More often than not, probing from a *single* system C doesn't cut it, instead behaviours comes in various shapes, each of which needs its own separate archetypal system:

$$B = \sum_{t \in T} \mathbf{Sys}(\mathsf{C}_t, -)$$

for a *family* $\mathsf{C} : T \longrightarrow \mathbf{Sys}$ (i.e. indexed by a *set* T).

Such functors are called **multirepresentable** (Karazeris and Velebil 2009).

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Such functors are called **multirepresentable** (Karazeris and Velebil 2009).

Remark

The family C is intended to be a *colax map of systems theories* from an **americ discrete theory** T:

(americ) its theory of compositions has only trivial compositions and

(discrete) its tight data is just a function.

Multirepresentable behaviour

More often than not, probing from a *single* system C doesn't cut it, instead behaviours comes in various shapes, each of which needs its own separate archetypal system:

$$B = \sum_{t \in T} \mathbf{Sys}(\mathsf{C}_t, -)$$

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However, the coproduct is take in the displayed category $[\mathbf{Sys}, \mathbf{Set}] \to [\mathbb{I}, \mathbf{Set}]$.

Multirepresentable behaviour: non-deterministic Moore machines

Example

The simplest case of multirepresentable behaviour is that of **runs** (or **paths**) of Moore machines. In that case we have

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\mathsf{T}} & \mathbf{Moore}_{\mathcal{P}}(\mathbf{Set}) \\ n & \longmapsto & 0 \rightsquigarrow \cdots \rightsquigarrow n \end{array}$$

where we stress T_n has interface $ny + 1$.

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On interfaces, a map $\mathsf{T}_n \to \mathsf{S}$ corresponds to a choice of $n+1$ outputs and n compatible inputs:

$$\binom{n}{n+1} \rightrightarrows \binom{I}{O} \quad \leftrightsquigarrow \quad \{((o_0, \ldots, o_n), (i_1, \ldots, i_n)) \mid i_{k+1} \in O_k \text{ for } 0 \leq k < n\}$$

and on systems, to a suitable sequence of n transitions $s_0 \overset{i_1}{\rightsquigarrow} s_1 \overset{i_2}{\rightsquigarrow} \cdots \overset{i_n}{\rightsquigarrow} s_n$ such that $v(s_k) = o_k$.

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Example

Likewise, the family L_n described before multirepresents the theory of loops.

Compositionality of multirepresentable behaviour

To construct the monoidal laxators, we need the family $\mathsf{C} : T \longrightarrow \mathbf{Sys}$ to be *colax monoidal*, thus

1. T must be a monoid
2. we must have coherent maps $\mathsf{C}_{s \otimes t} \longrightarrow \mathsf{C}_s \otimes \mathsf{C}_t$

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In any case we get (analogously on $\mathbb{I}$):

$$\sum_t \mathbf{Sys}(\mathsf{C}_t, \mathsf{S}) \times \sum_s \mathbf{Sys}(\mathsf{C}_s, \mathsf{R}) \xrightarrow{\sim} \sum_{t,s} \mathbf{Sys}(\mathsf{C}_t, \mathsf{S}) \times \mathbf{Sys}(\mathsf{C}_s, \mathsf{R}) \xrightarrow{\sum_{t,s} \nu_{s,t}} \sum_{t,s} \mathbf{Sys}(\mathsf{C}_{t\otimes s}, \mathsf{S}\otimes\mathsf{R}) \xrightarrow{\sum_\otimes} \sum_t \mathbf{Sys}(\mathsf{C}_t, \mathsf{S}\otimes\mathsf{R})$$

We don't have much control over $\sum_\otimes$ (invertible when $(t, s) \mapsto t \otimes s$ is—rarely).

Note: this is a pointwise coproduct but not a coproduct in $[\mathbf{Sys}, \mathbf{Set}]$!

Compositionality of multirepresentable behaviour

The absence of non-trivial loose arrows in the indexing T makes the compositional laxators analogous to the simply representable situation:

$$\sum_{t\in T} \mathbb{I}(H_t, p) \times \sum_{t\in T} \mathbb{I}(H_t, q) \cong \sum_{t\in T} \mathbb{I}(H_t, p) \times \mathbb{I}(H_t, q) \xrightarrow{\sum_{t\in T}(\boxminus)} \sum_{t\in T} \mathbb{I}(H_t, p \odot q)$$

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Theorem

The behaviour multirepresented by $\mathsf{C} : T \to \mathbf{Sys}$ is strongly compositional iff

1. *each H_t is spanlike,*
2. *each C_t is observable.*

Plurirepresentable behaviour

The family of systems we want to use to induce a theory of behaviour are not unrelated to each other, and thus rather than a multirepresentable functor we get a **plurirepresentable** one:

$$B = \mathrm{colim}_{t \in \mathbf{T}} \mathbf{Sys}(\mathsf{C}_t, -)$$

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for a *diagram* $\mathsf{C} : \mathbf{T} \to \mathbf{Sys}$.

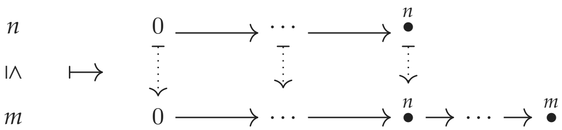
The morphisms of $\mathbf{T}$ make this situation particularly interesting, since they witness a 'geometric structure' on timepieces (especially when $\mathbf{T}$ is endowed with a coverage).

Plurirepresentable behaviour: non-deterministic Moore machines

Example

The theory of **maximal runs** of Moore machines is plurirepresented by

$$(\mathbb{N}, \leq) \longrightarrow \mathbf{Moore}_{\mathcal{P}}(\mathbf{Set})$$



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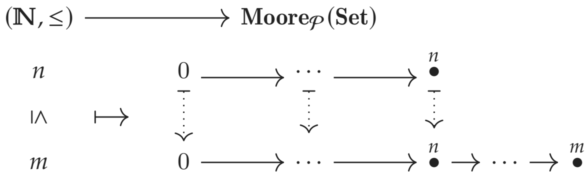

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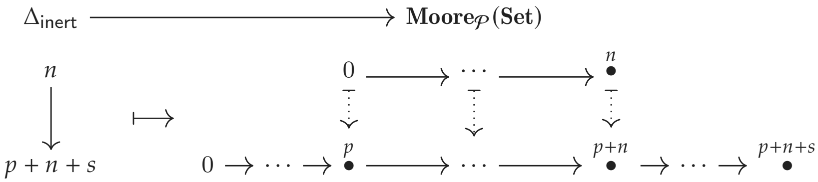
The theory of **bidirectional runs** is obtained by allowing time to extend backwards:



Plurirepresentable behaviour: non-deterministic Moore machines

Example

The family of loops L_n can be indexed by $(\mathbb{N}, \mid)^{\mathsf{op}}$, since a loop L_n can be winded up around a loop L_m only if $m \mid n$. Then $\operatorname{colim}_n \mathsf{L}_n\mathsf{S}$ yields the **minimal/indecomposable loops** in S and thus is less redundant than $\sum_n \mathsf{L}_n\mathsf{S}$. One can do better by categorifying...

Compositionality of plurirepresentable behaviour

Plurirepresentable behaviour has much of the same problems regarding monoidality as multirepresentable behaviour.

As for compositionality, we now need to assume $\mathbf{T}$ **is cofiltered** to get the right distributivity of colimit and pullback (since the colimit is indexed by $\mathbf{T}^{\mathsf{op}}$):

$$\mathrm{colim}_{t \in \mathbf{T}} \mathbb{I}(H_t, p) \mathbin{\underline{\times}} \mathrm{colim}_{t \in \mathbf{T}} \mathbb{I}(H_t, q) \overset{\sim}{\to} \mathrm{colim}_{t \in \mathbf{T}} \mathbb{I}(H_t, p) \mathbin{\underline{\times}} \mathbb{I}(H_t, q) \xrightarrow{\mathrm{colim}_{t \in \mathbf{T}} (\Xi)} \mathrm{colim}_{t \in \mathbf{T}} \mathbb{I}(H_t, p \odot q)$$

$$\mathrm{colim}_{t \in \mathbf{T}} \mathbf{Sys}(\mathsf{C}_t, \mathsf{S}) \mathbin{\underline{\times}} \mathrm{colim}_{t \in \mathbf{T}} \mathbb{I}(H_t, p) \overset{\sim}{\to} \mathrm{colim}_{t \in \mathbf{T}} \mathbf{Sys}(\mathsf{C}_t, \mathsf{S}) \mathbin{\underline{\times}} \mathbb{I}(H_t, p) \xrightarrow{\mathrm{colim}_{t \in \mathbf{T}} (\bullet)} \mathrm{colim}_{t \in \mathbf{T}} \mathbf{Sys}(\mathsf{C}_t, \mathsf{S} \bullet p)$$

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Theorem

The behaviour plurirepresented by $\mathsf{C} : \mathbf{T} \to \mathbf{Sys}$ *is strongly compositional iff*

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2. *each* H_t *is spanlike,*
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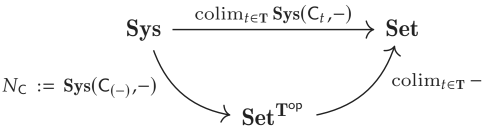
Better behaved behaviour: nerve behaviour

Given $\mathsf{C} : \mathbf{T} \longrightarrow \mathbf{Sys}$, we get a behaviour in $\mathbf{T}$-variable sets by a **nerve construction**:



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\mathbf{Sys} & \xrightarrow{\operatorname{colim}_{t \in \mathbf{T}} \mathbf{Sys}(\mathsf{C}_t, -)} & \mathbf{Set} \\
& {\scriptstyle N_{\mathsf{C}} := \mathbf{Sys}(\mathsf{C}_{(-)}, -)} \searrow \quad \nearrow {\scriptstyle \operatorname{colim}_{t \in \mathbf{T}} -} & \\
& \mathbf{Set}^{\mathbf{T}^{\mathrm{op}}} &
\end{array}
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Now considering $\mathbf{Set}^{\mathbf{T}^{\mathrm{op}}}$ with

1. ....pointwise cartesian products N_{C} **is strong monoidal iff each** C_t **is cartesian**.
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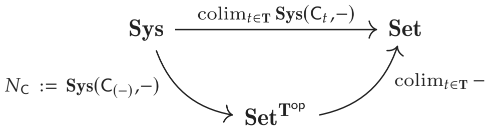

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$$N_{\mathsf{C}} := \mathbf{Sys}(\mathsf{C}_{(-)}, -) : \mathbf{Sys} \longrightarrow \mathbf{Set}^{\mathbf{T}^{\mathrm{op}}}, \qquad \operatorname{colim}_{t \in \mathbf{T}} - : \mathbf{Set}^{\mathbf{T}^{\mathrm{op}}} \longrightarrow \mathbf{Set}$$

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Theorem

The nerve behaviour $\mathbf{Sys}$ *induced by* $\mathsf{C} : \mathbf{T} \longrightarrow \mathbf{Sys}$ *is strong monoidal (wrt Day) and compositional iff*

1. C *is strong monoidal,*
2. *each* H_t *is spanlike,*
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3. Maps of timepieces (*time extensions*) can be used to define categorically the behavioural properties of systems.

   The most famous example is (Joyal, Nielsen, and Winskel 1996) using time-injective maps to define bisimulation. In (Baltieri, Biehl, Capucci, and Virgo 2025) we give a definition of 'model of a system' based on this.

Thanks!

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