Distillation systems as models of homotopy colimits

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What is homotopy theory?

Homotopy Theory is...

a branch of mathematics, particularly within algebraic topology, that studies continuous deformations (homotopies) of functions or mappings.

Google AI Overview Summary, May 1 2025

https://www.shapeways.com/product/6CJQ9GXWW/topology-joke

What is homotopy theory?

Let X and Y be topological spaces. A map between them is a continuous function.
Let $I = [0, 1]$ denote the unit interval.

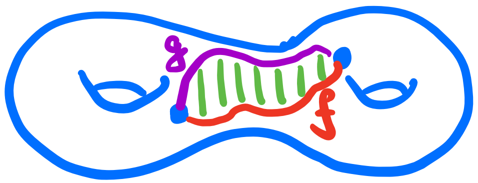
Definition

Two maps $f, g : X \to Y$ are homotopy equivalent if there exists a homotopy

$$H : X \times I \to Y$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. In this case we write $f \simeq g$.

Example: $X = I$ and $f: I \to Y$ and $g: I \to Y$ are paths:



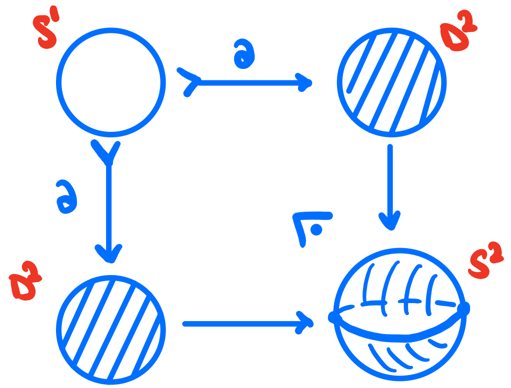
The image of a homotopy H fills the space between the paths f and g. It is the "movie" depicting a deformation of one path into the other.

We write $X \simeq Y$ when $\exists\, f : X \to Y$ and $g : Y \to X$ s.t. $fg \simeq 1_Y$ and $gf \simeq 1_X$.

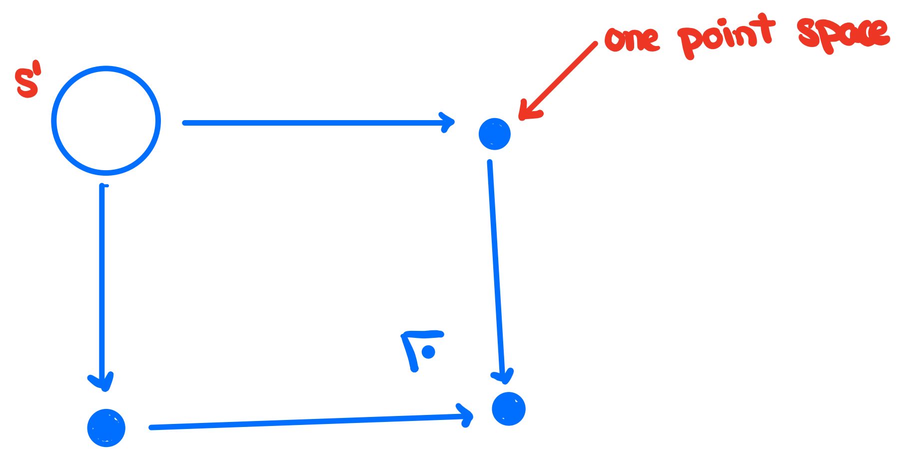
Motivating Example: homotopy pushouts

Problem: The strict pushout is not homotopy invariant.

Example: Two disks glued along a common boundary circle.



using the fact that $D^2 \simeq *$, can we rewrite this?



The pushouts are clearly not homotopy equivalent since $S^2 \not\simeq *$

Universal property of the colimit

Definition

Let $\mathcal{C}$ be any category, $\mathcal{I}$ be a small category. The colimit of $X : \mathcal{I} \to \mathcal{C}$ (if it exists) is the initial object in a category of cocones for X.

Example:

Let I be the category

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & 1 \\
2 &
\end{tikzcd}$$

The colimit of $X : I \longrightarrow \mathcal{C}$ is the pushout

$$\begin{tikzcd}
X(\emptyset) \arrow[r] \arrow[d] & X(1) \arrow[d] \arrow[ddr, bend left] & \\
X(2) \arrow[r] \arrow[drr, bend right] & X(1,2) \arrow[dr, dashed, "\exists!", "h"'] & \\
 & & Y
\end{tikzcd}$$

The universal property says the red cone is initial — i.e. $\exists!$ h as depicted.

Approach #1: Homotopy Colimits as a concept

There are two basic approaches to making colimits 'homotopical' in the literature.

Definition (Dwyer-Hirschorn-Kan-Smith)

A category $\mathcal{C}$ is a **homotopical category** if $\mathcal{C}$ contains a distinguished set W of morphisms that satisfy

- W contains all identity maps of $\mathcal{C}$
- W has the $2-of-6$ property, meaning that if the first and second composites are in W then so is each map and every composite:

$$\bullet \xrightarrow{r} \bullet \xrightarrow{s} \bullet \xrightarrow{t} \bullet.$$

The 2-of-6 property:

if $sr, ts \in W$ then $r, s, t, tsr \in W$.

Approach #1: Homotopy Colimits as a concept

The colimit is the initial object in a category of cocones for $X : \mathcal{I} \to \mathcal{C}$.
Can this be adapted?

Definition

The homotopically initial objects are defined by the property that the full subcategory spanned by them is empty or homotopically contractible.

Approach #1: Homotopy Colimits as a concept

The homotopy colimit is a homotopically initial object in a category of cocones for $X : \mathcal{I} \to \mathcal{C}$.

Definition

Homotopically initial objects are weakly equivalent up to a homotopically unique weak equivalence.

Problem: the concept of a homotopy colimit doesn't produce a construction of the homotopy colimit and allows for a lot of choices.

Approach #2: Homotopy Colimits as a construction

Definition (Quillen, Riehl)

A **model structure** on a complete and cocomplete category $\mathcal{C}$ consists of three classes of morphisms W, C and F such that

- $(C \cap W, F)$ and $(C, F \cap W)$ are weak factorization systems on $\mathcal{C}$ and
- W satisfies the 2-of-3 property.

Example:

The idea of a model structure on $\mathcal{C} = \mathrm{Top}$:

objects = topological spaces*
morph. = continuous functions
weak equivs = (weak) homotopy equiv.s*
cofibrations = inclusions*

* these items are oversimplified, more care is needed.

Factorization of f:

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "\simeq"'] & & b \\ & c \arrow[ur] & \end{tikzcd}$$

a cofibration and weak equiv $C \cap W$

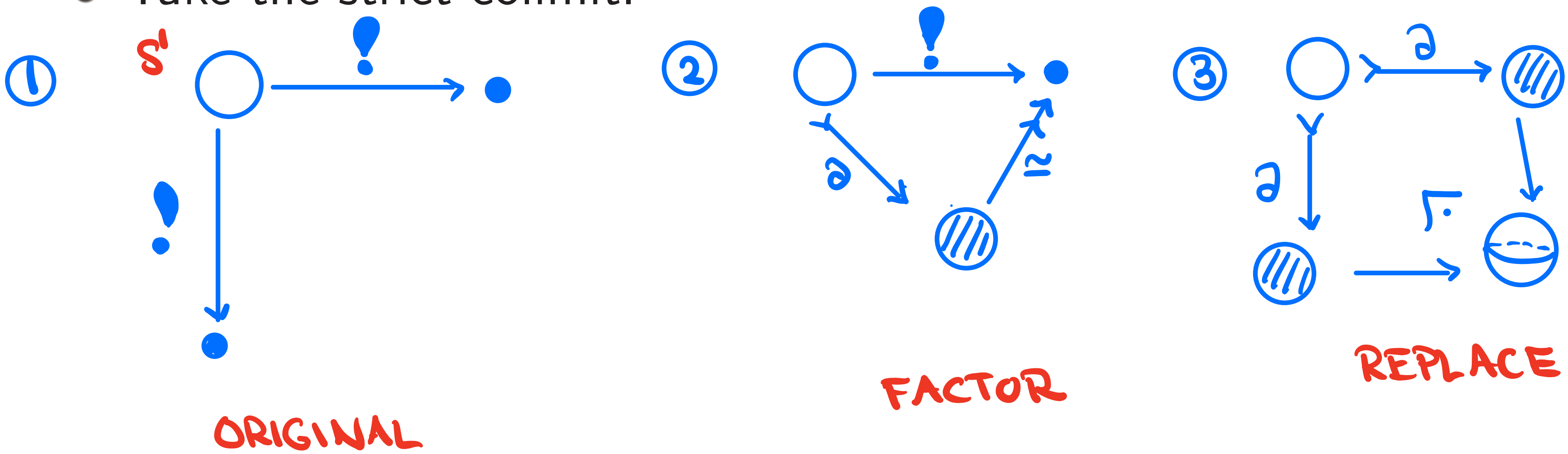
a fibration $\mathcal{F}$

There should be another factorization using $(C, \mathcal{F} \cap W)$.

Approach #2: Homotopy Colimits as a construction

In this case, a homotopy colimit is a procedure:

- Replace the morphisms in the diagram $X : \mathcal{I} \to \mathcal{C}$ by cofibrations up to weak equivalence,
- Take the strict colimit.



Problem: We don't always have a model category structure on hand, cofibrant replacement is not always functorial.

Middle ground

Let $\mathcal{C}$ be a category with a terminal object ∞.

Properties of homotopy colimits

1. Let $F : \mathbb{I} \times \mathbb{J} \to \mathcal{C}$, then $\mathrm{hocolim}_{\mathbb{I}}\, \mathrm{hocolim}_{\mathbb{J}}\, F \cong \mathrm{hocolim}_{\mathbb{I}\times\mathbb{J}}\, F$.
   AKA: Fubini property.
2. Let $\alpha : \mathbb{I} \to \mathbb{J}$ and $F : \mathbb{J} \to \mathcal{C}$, then $\mathrm{hocolim}_{\mathbb{I}}\, F \circ \alpha \to \mathrm{hocolim}_{\mathbb{J}}\, F$.
3. Let $\mathcal{C}$ be a basepointed category* then $\mathrm{hocolim}_{\mathbb{I}}\, cst_{\infty} = \infty$.
4. Let $P(0)$ be the trivial category then $\mathrm{hocolim}_{P(0)}\, F \to F(\emptyset)$.
5. If $F \simeq G$ (defined pointwise), $\mathrm{hocolim}_{\mathbb{I}}\, F \simeq \mathrm{hocolim}_{\mathbb{I}}\, G$.
   $F(i) \simeq G(i)\ \forall\, i \in \mathrm{ob}\, \mathbb{I}$

* In the live talk on May 1, I forgot to add the hypothesis that $\mathcal{C}$ has a basepoint - the terminal object is also initial.

Where do these properties come from?

Let $(\mathcal{A}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category

Definition

A (left) $\mathcal{A}$-actegory is a category $\mathcal{C}$ with a functor $- \bullet - : \mathcal{A} \times \mathcal{C} \to \mathcal{C}$ and two natural isomorphisms

- $\eta_x : x \xrightarrow{\cong} I \bullet x$
- $\mu_{a,b,x} : a \bullet (b \bullet x) \xrightarrow{\cong} (a \otimes b) \bullet x$

satisfying associativity and unit conditions.

Examples

Cat = category of small categories
CAT = category of all categories
} These are monoidal using the Cartesian product

<u>Actegory Structures</u>

1. The trivial structure $(\mathrm{CAT}, \mathrm{Triv})$

<u>Action</u>: $\mathrm{Cat}^{op} \times \mathrm{CAT} \xrightarrow{\pi_2} \mathrm{CAT}$ projection

$(\mathcal{I}, \mathcal{C}) \longmapsto \mathcal{C}$

<u>Unit</u>: $\eta_{\mathcal{C}} : \mathcal{C} \xrightarrow{=} \pi_2(\mathcal{I}, \mathcal{C}) = \mathcal{C}$ identity

<u>multiplication</u>: $\pi_2(\mathcal{I}, \pi_2(\mathcal{J}, \mathcal{C})) \xrightarrow{=} \pi_2(\mathcal{I} \times \mathcal{J}, \mathcal{C})$ identity

$= \pi_2(\mathcal{I}, \mathcal{C})$

$= \mathcal{C} \xrightarrow{id_{\mathcal{C}}} = \mathcal{C}$

The action is trivially unital and associative.

Examples

② The "Fun" action of Cat^{op} on CAT: (CAT, Fun)

Action: $\mathrm{Fun}: Cat^{op} \times CAT \longrightarrow CAT$

$(\mathcal{I}, \mathcal{C}) \longmapsto \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ = category of functors

multiplication: $\mu: \mathrm{Fun}(\mathcal{I}, \mathrm{Fun}(\mathcal{J}, \mathcal{C})) \xrightarrow{\cong} \mathrm{Fun}(\mathcal{I} \times \mathcal{J}, \mathcal{C})$

from the usual closed structure on CAT.

unit: $\eta_{\mathcal{C}}: \mathcal{C} \longrightarrow \mathrm{Fun}(P(\emptyset), \mathcal{C})$ where $P(\emptyset)$ is the unit of $\times$ on Cat

This is the functor taking $x \in \mathrm{ob}\,\mathcal{C}$ to $\mathrm{cst}_x: P(\emptyset) \longrightarrow \mathcal{C}$, the unique functor defined by $\mathrm{cst}_x(\emptyset) = x$.

Where do these properties come from?

Definition

Let $\mathcal{C}$ be a category and $\mathbb{D}$ be a 2-category with underlying category $\mathcal{D}$. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. An **oplax natural transformation** $\tau : F \Rightarrow G$ is

- for all $x \in \mathcal{C}$, $\tau_0(x) : F(x) \to G(x)$, and
- for all $f : x \to y$ in $\mathcal{C}$, a 2-cell $\tau_1(f)$:

NB: τ_0: ob $\mathcal{C} \longrightarrow \mathbb{D}_1$, a function

NB: τ_1: mor $\mathcal{C} \longrightarrow \mathbb{D}_2$, a function

$$\begin{tikzcd} F(x) \arrow[r, "\tau_0(x)"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\Nearrow"] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "\tau_0(y)"'] & G(y) \end{tikzcd}$$

which respect identity maps and composites.

Where do these properties come from?

Let $\mathcal{A}$ be a monoidal category, let $\mathcal{C}$ be an $\mathcal{A}$-actegory, and let $\mathbb{D}$ be a 2-category whose underlying category $\mathcal{D}$ is an $\mathcal{A}$-actegory.

Definition

A lax $\mathcal{A}$-linear morphism from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ together with an oplax natural transformation $\tau : \bullet_{\mathcal{D}} \circ F \to F \circ \bullet_{\mathcal{C}}$.

- $\tau_0(a, x) : a \bullet_{\mathcal{D}} F(x) \to F(a \bullet_{\mathcal{C}} x)$ for all $(a, x) \in \mathcal{A} \times \mathcal{C}$
- $\tau_1(\alpha, f)$ for all $(\alpha, f) \in \mathcal{A} \times \mathcal{C}$:

$$\begin{tikzcd}
a \bullet_{\mathcal{D}} F(x) \arrow[r, "\tau_0(a,x)"] \arrow[d, "\alpha \bullet_{\mathcal{D}} f"'] & F(a \bullet_{\mathcal{C}} x) \arrow[d, "F(\alpha \bullet_{\mathcal{C}} f)"] \\
b \bullet_{\mathcal{D}} F(y) \arrow[r, "\tau_0(b,y)"'] \arrow[ur, phantom, "\Nearrow \tau_1(\alpha, f)"] & F(b \bullet_{\mathcal{C}} y)
\end{tikzcd}$$

Where do these properties come from?

Definition

A **distillation system** on (Cat^{op}, CAT) consistes of a lax Cat^{op}-linear morphism

$$(Id, \delta, E, U) : (CAT, Triv) \to (CAT, Fun)$$

which is **pseudo-multiplicative** and **pseudo-unital**. + coherences

That is, it is a Cat^{op}-linear morphism $(CAT, Triv) \longrightarrow (CAT, Fun)$ whose underlying functor $CAT \longrightarrow CAT$ is id_{CAT}, which is unital and multiplicative up to isomorphism.
(invertible 2-cell)

The data of a distillation system

① There is an oplax natural transformation

$$\delta_0 : \mathrm{Fun}(\mathcal{I}, \mathrm{id}(\mathcal{C})) \longrightarrow \mathrm{id}(\pi_2(\mathcal{I},\mathcal{C}))$$
$$= \mathrm{Fun}(\mathcal{I},\mathcal{C}) \xrightarrow{\delta_0} \mathcal{C}$$

This models the homotopy colimit

② For $\alpha : \mathcal{I} \longrightarrow \mathcal{J}$ and $\phi : \mathcal{C} \longrightarrow \mathcal{D}$, there is a 2-cell

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{J},\mathcal{C}) \arrow[r, "\delta_0"] \arrow[d, "\mathrm{Fun}(\alpha,\phi)"'] \arrow[dr, phantom, "\delta_1(\alpha,\phi)"] & \mathcal{C} \arrow[d, "\phi"] \\
\mathrm{Fun}(\mathcal{I},\mathcal{D}) \arrow[r, "\delta_0"'] & \mathcal{D}
\end{tikzcd}$$

$\mathrm{hocolim}_{\mathcal{I}}\, \phi \circ F \circ \alpha \Longrightarrow \phi \circ \mathrm{hocolim}_{\mathcal{J}} F$

③ Pseudo-multiplicative:

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{I}, \mathrm{Fun}(\mathcal{J},\mathcal{C})) \arrow[r, "\mathrm{Fun}(\mathcal{I},\delta_0)"] \arrow[d, "\mu"', "\cong"] \arrow[dr, phantom, "\overset{E}{\swarrow}\cong"] & \mathrm{Fun}(\mathcal{I},\mathcal{C}) \arrow[d, "\delta_0(\mathcal{I},\mathcal{C})"] \\
\mathrm{Fun}(\mathcal{I}\times\mathcal{J},\mathcal{C}) \arrow[r, "\delta_0(\mathcal{I}\times\mathcal{J},\mathcal{C})"'] & \mathcal{C}
\end{tikzcd}$$

The data of a distillation system

④ Pseudo-unital:

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "M"] \arrow[dr, "\mathrm{id}_{\mathscr{C}}"'] & {} \arrow[d, phantom, "\overset{\cong}{\Swarrow} \mathcal{U}"] & \mathrm{Fun}(\mathcal{P}(\emptyset), \mathscr{C}) \arrow[dl, "S_0"] \\
& \mathscr{C} &
\end{tikzcd}$$

Middle ground

Let $\mathcal{C}$ be a category with a terminal object ∞.

Properties of homotopy colimits

1. Pseudo-multiplicativity:

$$\mathrm{hocolim}_{\mathbb{I}}\, \mathrm{hocolim}_{\mathbb{J}}\, F \cong \mathrm{hocolim}_{\mathbb{I}\times\mathbb{J}}\, F$$

2. Naturality of δ_1: (special case $\phi = id$)

$$\mathrm{hocolim}_{\mathbb{I}}\, F \circ \alpha \to \mathrm{hocolim}_{\mathbb{J}}\, F$$

3. Naturality of δ_1 and unitality*:

$$\mathrm{hocolim}_{\mathbb{I}}\, cst_{\infty} = \infty$$

4. Unitality:

$$\mathrm{hocolim}_{P(0)}\, F \to F(\emptyset)$$

* see earlier note: This only holds when $\mathcal{C}$ is basepointed.

Examples

- The conceptual definition of a homotopy colimit due to [DHKS] is *not* an example of a distillation system (properties only hold up to weak equivalence).
- Constructive definition of a homotopy colimit using model categories (e.g. Bousfield-Kan) are examples of distillation system.
- Other constructions of homotopy colimits - e.g. using the mapping cone to construct homotopy colimits in chain complexes - should also work.

THANK YOU!