Cartesian Fermat Categories*

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Quick Hello!

- Full Name: Jean-Simon Pacaud Lemay, please feel free to call me **JS**
- I'm from Québec, Canada (I'm currently in Québec, and actually giving this talk from my childhood bedroom!)
- I'm a lecturer/assistant professor at **Macquarie University** (Sydney, Australia)
- I'm a category theorist, and I study:

  • **Differential Categories**
  • Tangent Categories
  • Differential Geometry, Algebraic Geometry, Differential Algebras
  • Traced Monoidal Categories
  • Restriction Categories
  • Other stuff...

If you find differential categories interesting and would like to chat/work together or even visit our category theory group at Macquarie: feel free to come to talk to me or reach out by email!

Motivation – Cartesian Differential Categories

- **Cartesian differential categories** (CDC) is a categorical framework for the foundations of the multivariable differential calculus (over Euclidean spaces).
- CDC have been able to recapture lots of concept from differential calculus: linear maps, partial derivatives, differential equations, de Rham complex, etc.
- CDC are also very popular in computer science because they provide the categorical foundations of the differential lambda calculus, differentiable programming, and certain automatic differentiation techniques used for machine learning.

Some introductory references:

Blute, R., Cockett, R., Seely, R.A.G. **Cartesian Differential Categories** (2009)

Garner, R, and Lemay, J-S P. **Cartesian differential categories as skew enriched categories**.

Manzonetto, G. **What is a Categorical Model of the Differential and the Resource λ-Calculi?**. (2012)

Motivation – Fermat Theories

- Another approach for the categorical foundations of differential calculus is via:

Fermat Theories

E. Dubuc, A. Kock, **On 1-form classifiers**

`https://ncatlab.org/nlab/show/Fermat+theory`

- Briefly, a Fermat theory is a Lawvere theory that is an extension of commutative rings (so we can add and multiply maps) that adds a stronger version **Hadamard's lemma** as an axiom. The main idea is that for every map $f(x)$ there exists a **unique** map $\tilde{f}(x, y)$ such that:

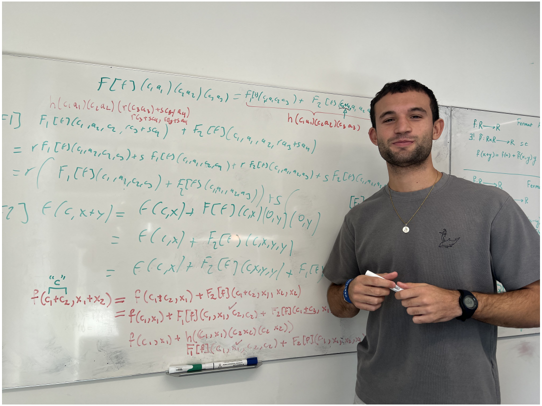
$$f(x + y) = f(x) + \tilde{f}(x, y)y$$

From here, one can define the derivative of f as $f'(x) = \tilde{f}(x, 0)$.

- Ben MacAdam (an ex-grad student of Robin Cockett at the same time that I was) showed that every Fermat theory is a CDC (presented at FMCS2016 in Vancouver, and also in unpublished notes).

- So Fermat theories are CDC. Great! However, in general though: 1) CDC are not Lawvere theories and 2) we can't always multiply maps in a CDC...

Motivation – Enter Carlos



- Carlos was my undergrad student at Macquarie who just wanted to learn about doing math research. He chose to learn about Fermat theories and how they were CDC (because I had never really sat down and looked at Fermat theories and Ben's proof in detail). We made quick progress, and early on we figured out how to properly generalize Fermat theories...
- **TODAY'S STORY**: **Cartesian Fermat categories**, the CDC analogue of a Fermat theory.
- **MAIN RESULT**: Every Cartesian Fermat category is a CDC.

Disclaimer

- I've recently been made aware that Ben (with Robin and Jonathan) probably thought about some of these ideas (though unpublished).
- So Carlos and I may have rediscovered somethings Ben did...
- I've reached out to Ben to compare ideas.

Cartesian Left k-Linear Categories – Definition

The underlying category of a Cartesian differential/Fermat category is a Cartesian left k-linear category, which is a category where we can take sums and scalar multiplication of maps, but where not every map is k-linear.

Definition

For a commutative semiring k, a **left k-linear category** is a category $\mathbb{X}$ such that each homset $\mathbb{X}(A, B)$ is a k-module with:

$$+ : \mathbb{X}(A, B) \times \mathbb{X}(A, B) \to \mathbb{X}(A, B) \qquad 0 \in \mathbb{X}(A, B) \qquad \cdot : k \times \mathbb{X}(A, B) \to \mathbb{X}(A, B)$$

such that pre-composition preserves the k-linear structure:

$$(s \cdot f + t \cdot g) \circ x = s \cdot (f \circ x) + t \cdot (g \circ x)$$

A map $f : A \to B$ is said to be **k-linear** if post-composition by f preserves the k-linear structure:

$$f \circ (s \cdot x + t \cdot y) = s \cdot (f \circ x) + t \cdot (f \circ y)$$

A **Cartesian left k-linear category** (CLLC$_k$) is a left k-linear category with finite products such that the projection maps $\pi_j : A_1 \times \ldots \times A_n \to A_j$ are k-linear.

Cartesian Left k-Linear Categories – Examples

Example

A **Cartesian k-linear category** is a CLLC_k such that every map is k-linear. Equivalently, it is a category with finite products that is enriched over k-modules. In this case, this product $\times$ is in fact a biproduct! So a Cartesian k-linear category is equivalently a category with finite biproducts that is also enriched over k-modules. For example, MOD_k is a Cartesian k-linear category.

Example

Let Poly_k be the Lawvere theory of polynomials, that is, the category whose objects are $n \in \mathbb{N}$ and where a map $P : n \to m$ is a tuple of polynomials:

$$P = \langle p_1(\vec{x}), \ldots, p_m(\vec{x}) \rangle \qquad p_i(\vec{x}) \in R[x_1, \ldots, x_n]$$

Poly_k is a CLLC_k (where $n \times m = n + m$). The k-linear maps in Poly_k are tuples of polynomials which are k-linear.

Example

Let SMOOTH be the category of smooth real functions, that is, the category whose objects are the Euclidean vector spaces $\mathbb{R}^n$ and whose maps are smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, which is actually an m-tuple of smooth functions:

$$F = \langle f_1, \ldots, f_m \rangle \qquad f_i : \mathbb{R}^n \to \mathbb{R}$$

SMOOTH is a $\mathsf{CLLC}_{\mathbb{R}}$. Note that $\mathsf{Poly}_{\mathbb{R}}$ is a sub-$\mathsf{CLLC}_{\mathbb{R}}$ of SMOOTH. The $\mathbb{R}$-linear maps in SMOOTH are precisely the usual $\mathbb{R}$-linear morphisms $F : \mathbb{R}^n \to \mathbb{R}^m$.

Cartesian Differential Categories – Definition

Definition

A **Cartesian** k**-differential category** (CDC$_k$) is a Cartesian left k-linear category $\mathbb{X}$ equipped with a **differential combinator** D, which is a family of operators:

$$\mathsf{D} : \mathbb{X}(A, B) \to \mathbb{X}(A \times A, B) \qquad \frac{f : A \to B}{\mathsf{D}[f] : A \times A \to B}$$

where $\mathsf{D}[f]$ is called the **derivative** of f, and which satisfies seven axioms which capture the basics of the derivative from differential calculus (which we will see in a few slides).

Cartesian Differential Categories – Main Example

Example

SMOOTH is a CDC$_{\mathbb{R}}$, where for a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, its derivative $\mathsf{D}[F] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is:

$$\mathsf{D}[F](\vec{x}, \vec{y}) := \mathbf{J}(F)(\vec{x}) \cdot \vec{y} = \left\langle \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(\vec{x}) y_i, \ldots, \sum_{i=1}^{n} \frac{\partial f_m}{\partial x_i}(\vec{x}) y_i \right\rangle$$

where $\mathbf{J}(F)(\vec{x})$ is the Jacobian of F at $\vec{x}$ and where $\cdot$ is matrix multiplication. This is the *total* derivative of F.

In the special case of $m = 1$, so for a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, its derivative $\mathsf{D}[F] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is:

$$\mathsf{D}[f](\vec{x}, \vec{y}) = \nabla(f)(\vec{x}) \cdot \vec{y} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}) y_i$$

where $\nabla(f)$ is the gradient of f.

Cartesian Differential Categories – Other Examples

Example

Any Cartesian k-linear category is a CDC_k, where for a map $f : A \to B$:

$$\mathsf{D}[f] := A \times A \xrightarrow{\pi_2} A \xrightarrow{f} B$$

For example, MOD_k is a CDC_k where $\mathsf{D}[f](x, y) = f(y)$.

Example

POLY_k is a CDC_k where for a map $P : n \to m$ with $P = \langle p_1, \ldots, p_n \rangle$, its derivative $\mathsf{D}[P] : n \times n \to m$ is:

$$\mathsf{D}[P] := \left\langle \sum_{i=1}^{n} \frac{\partial p_1}{\partial x_i} y_i, \ldots, \sum_{i=1}^{n} \frac{\partial p_n}{\partial x_i} y_i \right\rangle \qquad \sum_{i=1}^{n} \frac{\partial p_1}{\partial x_i} y_i \in R[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

In particular when $m = 1$, $p : n \to 1$ which is a polynomial $p(x_1, \ldots, x_n)$, its derivative $\mathsf{D}[p] : n \times n \to 1$ is the polynomial:

$$\mathsf{D}[p](x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} \frac{\partial p_1}{\partial x_i} y_i$$

Note that $\mathsf{POLY}_{\mathbb{R}}$ is a sub-$\mathsf{CDC}_{\mathbb{R}}$ of SMOOTH.

Differential Combinator – Axioms

To help us with the axioms for a differential combinator, we will use the following notation/proto-term logic:

$$\mathrm{D}[f](a, b) := \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b$$

Example

The notation comes from SMOOTH: $\mathrm{D}[f](\vec{x}, \vec{y}) := \nabla(f)(\vec{x}) \cdot \vec{y}$.

Remark

There is a sound and complete term logic for Cartesian differential categories. In short: anything we can prove using the term logic, holds in any Cartesian differential category. So doing proofs in the term logic is super useful!

CD.1 - k-Linearity of Combinator & CD.2 - k-Linearity in Second Argument

- k-linearity of Combinator:

$$\mathsf{D}[r \cdot f + s \cdot g] = r \cdot \mathsf{D}[f] + s \cdot \mathsf{D}[g]$$

$$\frac{\mathrm{d}r \cdot f(x) + s \cdot g(x)}{\mathrm{d}x}(a) \cdot b = r \cdot \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b + s \cdot \frac{\mathrm{d}g(x)}{\mathrm{d}x}(a) \cdot b$$

- k-linearity in Second Argument

$$\mathsf{D}[f] \circ \langle a, r \cdot b + s \cdot c \rangle = r \cdot \mathsf{D}[f] \circ \langle a, b \rangle + s \cdot \mathsf{D}[f] \circ \langle a, c \rangle$$

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot (r \cdot b + s \cdot c) = r \cdot \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b + s \cdot \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot c$$

CD.3 - Identities + Projections & CD.4 - Pairings

- Identities + Projections

$$\mathrm{D}[1] = \pi_1 \qquad \mathrm{D}[\pi_j] = \pi_{n+j}$$

$$\frac{\mathrm{d}x}{\mathrm{d}x}(a) \cdot b = b \qquad \frac{\mathrm{d}x_i}{\mathrm{d}(x_1, \ldots, x_n)}(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = b_j$$

- Pairings

$$\mathrm{D}[\langle f_1, \ldots, f_n \rangle] = \langle \mathrm{D}[f_1], \ldots, \mathrm{D}[f_n] \rangle$$

$$\frac{\mathrm{d}\langle f_1(x), \ldots, f_n(x) \rangle}{\mathrm{d}x}(a) \cdot b = \left\langle \frac{\mathrm{d}f_1(x)}{\mathrm{d}x}(a) \cdot b, \ldots, \frac{\mathrm{d}f_n(x)}{\mathrm{d}x}(a) \cdot b \right\rangle$$

Example

In SMOOTH, if $F = \langle f_1, \ldots, f_n \rangle$, then $\mathrm{D}[F](\vec{x}, \vec{y}) := \langle \mathrm{D}[f_1](\vec{x}, \vec{y}), \ldots, \mathrm{D}[f_n](\vec{x}, \vec{y}) \rangle$.

CD.5 - Chain Rule

Chain Rule:

$$\mathrm{D}[g \circ f] = \mathrm{D}[g] \circ \langle f \circ \pi_0, \mathrm{D}[f] \rangle$$

$$\frac{\mathrm{d}g(f(x))}{\mathrm{d}x}(a) \cdot b = \frac{\mathrm{d}g(y)}{\mathrm{d}y}(f(a)) \cdot \left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b \right)$$

CD.6 - Linearity in Second Argument & CD.7 - Symmetry

$$\cfrac{\cfrac{f : A \to B}{\mathrm{D}[f] : A \times A \to B}}{\mathrm{D}\left[\mathrm{D}[f]\right] : (A \times A) \times (A \times A) \to B}$$

- D-Linearity in Second Argument:

$$\mathrm{D}\left[\mathrm{D}[f]\right] \circ \langle a, 0, 0, b \rangle = \mathrm{D}[f] \circ \langle a, b \rangle$$

$$\frac{\mathrm{d}\frac{\mathrm{d}f(x)}{\mathrm{d}x}(y) \cdot z}{\mathrm{d}(y, z)}(a, 0) \cdot (0, b) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b$$

- Symmetry of Partial Derivatives:

$$\mathrm{D}\left[\mathrm{D}[f]\right] \circ \langle \langle a, b \rangle, \langle c, d \rangle \rangle = \mathrm{D}\left[\mathrm{D}[f]\right] \circ \langle \langle a, c \rangle, \langle b, d \rangle \rangle$$

$$\frac{\mathrm{d}\frac{\mathrm{d}f(x)}{\mathrm{d}x}(y) \cdot z}{\mathrm{d}(y, z)}(a, b) \cdot (c, d) = \frac{\mathrm{d}\frac{\mathrm{d}f(x)}{\mathrm{d}x}(y) \cdot z}{\mathrm{d}(y, z)}(a, c) \cdot (b, d)$$

Cartesian Differential Categories – Definition

Definition

A **Cartesian k-differential category** (CDC$_k$) is a Cartesian left k-linear category $\mathbb{X}$ equipped with a **differential combinator** D, which is a family of operators:

$$\mathsf{D} : \mathbb{X}(A, B) \to \mathbb{X}(A \times A, B) \qquad \frac{f : A \to B}{\mathsf{D}[f] : A \times A \to B}$$

where $\mathsf{D}[f]$ is called the **derivative** of f, and which satisfies **[CD.1]** to **[CD.7]**.

Remark

Differential combinators are not necessarily unique! Some CLLC$_k$ can have multiple differential combinators.

D-Linear Maps

In a CDC$_k$, a map $f : A \to B$ is said to be D-linear if:

$$\mathsf{D}[f] := A \times A \xrightarrow{\pi_2} A \xrightarrow{f} B \qquad\qquad \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b = f(b)$$

We can think of these as the degree 1 maps.

Example

In SMOOTH$_{\mathbb{R}}$, a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$ is D-linear in the Cartesian differential sense precisely when it is $\mathbb{R}$-linear in the classical sense.

- Every D-linear map is k-linear. **HOWEVER**:, the converse is not necessarily true. There are examples with k-linear maps that are not D-linear. More on this later...

What else can we do with Cartesian differential categories?

- Partial derivatives/differentiation in context (the simple slice of a CDC is a CDC).

- Study and solve differential equations, and also study exponential functions, trigonometric functions, hyperbolic functions, etc.

  - Cockett, R., Cruttwel, G., Lemay, J-S. P., **Differential equations in a tangent category I: Complete vector fields, flows, and exponentials**.
  - Lemay, J-S.P., **Exponential Functions for Cartesian Differential Categories**.

- Linearization, Jacobians and gradients:

  - Cockett, R., Lemay, J-S.P., **Linearizing Combinators**.
  - Lemay, J-S.P., **Jacobians and Gradients for Cartesian Differential Categories**.

- Foundations for automatic differentiation and machine learning algorithms via **reverse differentiation**.

  - Cockett, R., Cruttwell, G., Gallagher, J., Lemay, J.-S. P., MacAdam, B., Plotkin, G., & Pronk, D. (2020). **Reverse derivative categories.**
  - Wilson, P., & Zanasi, F. Reverse **Derivative Ascent: A Categorical Approach to Learning Boolean Circuits.**
  - Cruttwell, G., Gallagher, J., & Pronk, D. **Categorical semantics of a simple differential programming language.**
  - Cruttwell, G., Gavranovic, B., Ghani, N., Wilson, P., & Zanasi, F. **Categorical Foundations of Gradient-Based Learning.**

Cartesian Differential Categories – Even More Examples

Example

- The coKleisli category of a differential category is a CDC
  - R. Blute, R. Cockett, R.A.G. Seely, **Differential Categories**
- Every model of the differential λ-calculus induces a CDC.
  - Manzonetto, G., 2012. **What is a Categorical Model of the Differential and the Resource λ-Calculi?**.
- The differential objects of a Cartesian tangent category is a CDC
  - R. Cockett, G. Cruttwell **Differential structure, tangent structure, and SDG**
- Bauer et. al (BJORT) constructed an Abelian functor calculus model of a CDC.
  - Bauer, K., Johnson, B., Osborne, C., Riehl, E. and Tebbe, A., 2018. **Directional derivatives and higher order chain rules for abelian functor calculus.**
- There exists both free and cofree constructions of CDCs.
  - Cockett, J.R.B. and Seely, R.A.G., 2011. **The Faa di bruno construction**.

Fermat Theories are CDCs

Theorem (B. MacAdam)

A Fermat theory is a CDC.

Example

Examples of Fermat theories include:

- SMOOTH (the induced differential combinator from above thm is the one given previously)
- POLY_k (the induced differential combinator from above thm is the one given previously)
- Lots more examples can be found in:

Carchedi, D. & Roytenberg, D. **On Theories of Superalgebras of Differentiable Functions**.

Goal: give generalized version of Fermat theories, that are more of a direct analogue of CDC.

Cartesian Fermat Categories – Definition

Recall that the main idea about Fermat theories was that: for every map $f(x)$ there exists a **unique** map $\tilde{f}(x, y)$ such that:

$$f(x + y) = f(x) + \tilde{f}(x, y)y$$

But recall that I never mentioned multiplication in a CDC, so we need to fix that...

So instead the idea is that for every map $f(x)$ there exists a **unique** map $\mathsf{F}[f](x, y, z)$ such that:

$$f(x + y) = f(x) + \mathsf{F}[f](x, y, y)$$

where $\mathsf{F}[f](x, y, z) = \tilde{f}(x, y)z$

Definition

A **Cartesian k-Fermat category** (CFC_k) is a Cartesian left k-linear category $\mathbb{X}$ such that for every map $f : A \to B$, there exists a **unique** map $\mathsf{F}[f] : A \times A \times A \to B$ such that:

- **[F.1]** $\mathsf{F}[f]$ is k-linear in its third argument:

$$\mathsf{F}[f] \circ \langle x, y, r \cdot z + s \cdot w\rangle = r \cdot \mathsf{F}[f] \circ \langle x, y, z\rangle + s \cdot \mathsf{F}[f] \circ \langle x, y, w\rangle$$

- **[F.2]** $f \circ (x + y) = f \circ x + \mathsf{F}[f] \circ \langle x, y, y\rangle$

We call $\mathsf{F}[f]$ the **Fermat extension** of f (or just Fermat of f for short), and call $\mathsf{F}[-]$ the **Fermat combinator**.

Remark

For a CLLC_k being a CFC_k is a property!

Cartesian Fermat Categories – Examples

Example

SMOOTH is a CFC$_k$. For a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, $\mathsf{F}[f] : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as:

$$\mathsf{F}[f](\vec{x}, \vec{y}, \vec{z}) = \int_0^1 \nabla(f)(\vec{x} + t\vec{y}) \cdot \vec{z} \, \mathrm{d}t$$

We can see this as a generalization of Hadamard's Lemma, or an application of the Fundamental Theorem of Line Integration. More generally for $F : \mathbb{R}^n \to \mathbb{R}^m$, $\mathsf{F}[F] : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is:

$$\mathsf{F}[F](\vec{x}, \vec{y}, \vec{z}) = \langle \mathsf{F}[f_1](\vec{x}, \vec{y}, \vec{z}), \ldots, \mathsf{F}[f_n](\vec{x}, \vec{y}, \vec{z}) \rangle$$

Example

Every Cartesian k-linear category is a CFC$_k$ where for a map $f : A \to B$,

$$\mathsf{F}[f] := A \times A \times A \xrightarrow{\pi_3} A \xrightarrow{f} B$$

For example, MOD$_k$ is a CFC$_k$ where $\mathsf{F}[f](x, y, z) = f(z)$. This is a non Fermat theory example, since we cannot multiply!

Main Theorem: Cartesian Fermat Categories are CDCs

Theorem

A CFC_k is a CDC_k.

So from the Fermat combinator $\mathsf{F}[-]$ we can build a differential combinator $\mathsf{D}[-]$.

To explain this, let's first look at some useful identities of $\mathsf{F}[-]$ that we need for the differential combinator axioms.

For simplicity, I will use element/function notation.

Cartesian Fermat Categories – Chain Rule

Proposition

$\mathsf{F}[g \circ f](x, y, z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z)\right)$

Proof.

Need to show that $h(x, y, z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z)\right)$ satisfies **[F.1]** and **[F.2]**. $\square$

Cartesian Fermat Categories – Chain Rule

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[F.1]:

$$h(x, y, r \cdot z + s \cdot w) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, r \cdot z + s \cdot w)\right)$$

□

Proposition

$\mathsf{F}[g \circ f](x, y, z) = \mathsf{F}[g]\,(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z))$

Proof.

We need to show that $h(x, y, z) = \mathsf{F}[g]\,(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z))$ satisfies **[F.1]** and **[F.2]**.

[F.1]:

$$\begin{aligned} h(x, y, r \cdot z + s \cdot w) &= \mathsf{F}[g]\,(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, r \cdot z + s \cdot w)) \\ &= \mathsf{F}[g]\,(f(x), \mathsf{F}[f](x, y, y), r \cdot \mathsf{F}[f](x, y, z) + s \cdot \mathsf{F}[f](x, y, w)) \end{aligned}$$

$\square$

Cartesian Fermat Categories – Chain Rule

Proposition

$\mathsf{F}[g \circ f](x, y, z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z)\right)$

Proof.

We need to show that $h(x, y, z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z)\right)$ satisfies **[F.1]** and **[F.2]**.

[F.1]:

$$\begin{aligned}
h(x, y, r \cdot z + s \cdot w) &= \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, r \cdot z + s \cdot w)\right) \\
&= \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), r \cdot \mathsf{F}[f](x, y, z) + s \cdot \mathsf{F}[f](x, y, w)\right) \\
&= r \cdot \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z)\right) \\
&\quad + s \cdot \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, w)\right) \\
&= r \cdot h(x, y, z) + s \cdot h(x, y, w)
\end{aligned}$$

□

Proposition

$\mathsf{F}[g \circ f](x,y,z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x,y,y), \mathsf{F}[f](x,y,z)\right)$

Proof.

We need to show that $h(x,y,z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x,y,y), \mathsf{F}[f](x,y,z)\right)$ satisfies **[F.1]** and **[F.2]**.

[F.2]:

$$(g \circ f)(x+y) = \ g(f(x+y))$$

$\square$

Cartesian Fermat Categories – Chain Rule

Proposition

$\mathsf{F}[g \circ f](x,y,z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x,y,y), \mathsf{F}[f](x,y,z)\right)$

Proof.

We need to show that $h(x,y,z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x,y,y), \mathsf{F}[f](x,y,z)\right)$ satisfies **[F.1]** and **[F.2]**.

[F.2]:

$$\begin{aligned}(g \circ f)(x+y) &= g(f(x+y)) \\ &= g\left(f(x) + \mathsf{F}[f](x,y,y)\right)\end{aligned}$$

□

Cartesian Fermat Categories – Chain Rule

Proposition

$\mathsf{F}[g \circ f](x, y, z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z)\right)$

Proof.

We need to show that $h(x, y, z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z)\right)$ satisfies **[F.1]** and **[F.2]**.

[F.2]:

$$\begin{aligned}
(g \circ f)(x + y) = {} & g(f(x + y)) \\
= {} & g\left(f(x) + \mathsf{F}[f](x, y, y)\right) \\
= {} & g(f(x)) + \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, y)\right) \\
= {} & (g \circ f)(x) + h(x, y, y)
\end{aligned}$$

□

Cartesian Fermat Categories – Chain Rule

Proposition

$\mathsf{F}[g \circ f](x, y, z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z)\right)$

Proof.

So we have shown that $h(x, y, z) = \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z)\right)$ satisfies **[F.1]** and **[F.2]**, so by uniqueness of $\mathsf{F}[-]$, we get that $h(x, y, z) = \mathsf{F}[g \circ f](x, y, z)$. □

Cartesian Fermat Categories – k-Linear Structure

Lemma

$\mathsf{F}[r \cdot f + s \cdot g] = r \cdot \mathsf{F}[f] + s \cdot \mathsf{F}[g]$

Cartesian Fermat Categories – k-linear maps

Proposition

A map f is k-linear if and only if $\mathsf{F}[f](x, y, z) = f(z)$.

Lemma

$\mathsf{F}[1_A](x, y, z) = z$

Lemma

$\mathsf{F}[\pi_j](\vec{x}, \vec{y}, \vec{z}) = z_j$

Cartesian Fermat Categories – Pairings

Lemma

$\mathsf{F}\left[\langle f_1, \ldots, f_n \rangle\right] = \langle \mathsf{F}[f_1], \ldots, \mathsf{F}[f_n] \rangle$

Cartesian Fermat Categories – Higher-Order

$$\dfrac{\dfrac{f : A \to B}{\mathsf{F}[f] : A \times A \times A \to B}}{\mathsf{F}\left[\mathsf{F}[f]\right] : (A \times A \times A) \times (A \times A \times A) \times (A \times A \times A) \to B}$$

Lemma

$\mathsf{F}[f](x, y, z) = \mathsf{F}\left[\mathsf{F}[f]\right]((x, y, 0), (0, 0, y), (0, 0, z))$

Lemma

$\mathsf{F}\left[\mathsf{F}[f]\right]((x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)) = \mathsf{F}\left[\mathsf{F}[f]\right]((x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3))$

Main Theorem: Cartesian Fermat Categories are CDCs

Theorem

A CFC_k is a CDC_k where the differential combinator is defined as follows:

$$\mathsf{D}^{\mathsf{F}}[f] := \mathsf{F}[f] \circ \langle \pi_1, 0, \pi_2 \rangle$$

or in element notation:

$$\mathsf{D}^{\mathsf{F}}[f](x, y) = \mathsf{F}[f](x, 0, y)$$

Chain Rule Proof

Proof.

We want to show that $\mathsf{D}^{\mathsf{F}}[g \circ f](x, y) = \mathsf{D}^{\mathsf{F}}[g]\left(f(x), \mathsf{D}^{\mathsf{F}}[f](x, y)\right)$. $\square$

Chain Rule Proof

Proof.

We want to show that $\mathrm{D}^{\mathrm{F}}[g \circ f](x, y) = \mathrm{D}^{\mathrm{F}}[g]\left(f(x), \mathrm{D}^{\mathrm{F}}[f](x, y)\right)$.

$$\mathrm{D}^{\mathrm{F}}[g \circ f](x, y) = \mathrm{F}[g \circ f](x, 0, y)$$

$\square$

Chain Rule Proof

Proof.

We want to show that $\mathsf{D}^{\mathsf{F}}[g \circ f](x, y) = \mathsf{D}^{\mathsf{F}}[g]\left(f(x), \mathsf{D}^{\mathsf{F}}[f](x, y)\right)$.

$$\begin{aligned} \mathsf{D}^{\mathsf{F}}[g \circ f](x, y) = & \; \mathsf{F}[g \circ f](x, 0, y) \\ = & \; \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, 0, 0), \mathsf{F}[f](x, 0, y)\right) \end{aligned}$$

$\square$

Chain Rule Proof

Proof.

We want to show that $\mathsf{D}^{\mathsf{F}}[g \circ f](x, y) = \mathsf{D}^{\mathsf{F}}[g]\left(f(x), \mathsf{D}^{\mathsf{F}}[f](x, y)\right)$.

$$\begin{aligned}
\mathsf{D}^{\mathsf{F}}[g \circ f](x, y) &= \mathsf{F}[g \circ f](x, 0, y) \\
&= \mathsf{F}[g]\,(f(x), \mathsf{F}[f](x, 0, 0), \mathsf{F}[f](x, 0, y)) \\
&= \mathsf{F}[g]\,(f(x), 0, \mathsf{F}[f](x, 0, y))
\end{aligned}$$

$\square$

Chain Rule Proof

Proof.

We want to show that $\mathsf{D}^{\mathsf{F}}[g \circ f](x, y) = \mathsf{D}^{\mathsf{F}}[g]\left(f(x), \mathsf{D}^{\mathsf{F}}[f](x, y)\right)$.

$$\begin{aligned}
\mathsf{D}^{\mathsf{F}}[g \circ f](x, y) &= \mathsf{F}[g \circ f](x, 0, y) \\
&= \mathsf{F}[g]\left(f(x), \mathsf{F}[f](x, 0, 0), \mathsf{F}[f](x, 0, y)\right) \\
&= \mathsf{F}[g]\left(f(x), 0, \mathsf{F}[f](x, 0, y)\right) \\
&= \mathsf{F}[g]\left(f(x), 0, \mathsf{D}^{\mathsf{F}}[f](x, y)\right)
\end{aligned}$$

$\square$

Chain Rule Proof

Proof.

We want to show that $\mathsf{D}^{\mathsf{F}}[g \circ f](x, y) = \mathsf{D}^{\mathsf{F}}[g]\left(f(x), \mathsf{D}^{\mathsf{F}}[f](x, y)\right)$.

$$
\begin{aligned}
\mathsf{D}^{\mathsf{F}}[g \circ f](x, y) = \; & \mathsf{F}[g \circ f](x, 0, y) \\
= \; & \mathsf{F}[g]\,(f(x), \mathsf{F}[f](x, 0, 0), \mathsf{F}[f](x, 0, y)) \\
= \; & \mathsf{F}[g]\,(f(x), 0, \mathsf{F}[f](x, 0, y)) \\
= \; & \mathsf{F}[g]\left(f(x), 0, \mathsf{D}^{\mathsf{F}}[f](x, y)\right) \\
= \; & \mathsf{D}^{\mathsf{F}}[g]\left(f(x), \mathsf{D}^{\mathsf{F}}[f](x, y)\right)
\end{aligned}
$$

□

The other axioms follow from the analogue other identities.

Main Theorem: Cartesian Fermat Categories are CDCs

Theorem

A CFC_k is a CDC_k where the differential combinator is defined as follows:

$$\mathsf{D}^{\mathsf{F}}[f] := \mathsf{F}[f] \circ \langle \pi_1, 0, \pi_2 \rangle$$

or in element notation:

$$\mathsf{D}^{\mathsf{F}}[f](x, y) = \mathsf{F}[f](x, 0, y)$$

Example

- For a Cartesian k-linear category, $\mathsf{D}^{\mathsf{F}}[f](x, y) = f(y)$ as before.
- For SMOOTH, the induced differential combinator is precisely the one given before.

There is an issue...

Unfortunately, not every Fermat theory is a Cartesian Fermat category...

Thanks to the binomial theorem, POLY_k is always a Fermat theory.

Example

Let $k = \mathbb{Z}_2$ and consider $\mathsf{POLY}_{\mathbb{Z}_2}$. Consider the polynomial $p(x) = x^2$, which is in fact a $\mathbb{Z}_2$-linear map.

Consider the polynomials $q(x, y, z) = z^2$ and $r(x, y, z) = yz$, which are both $\mathbb{Z}_2$-linear in their third argument.

However, since:

$$q(x, y, y) = r(x, y, y) = y^2$$

we get that:

$$p(x + y) = p(x) + q(x, y, y) = p(x) + r(x, y, y)$$

Thus there are two distinct maps satisfying **[F.1]** and **[F.2]** for $p(x) = x^2$. So $\mathsf{POLY}_{\mathbb{Z}_2}$ is not a CFC_k but is still a CDC_k.

So maybe Cartesian Fermat categories are not the right name...

The issue is with D^{F}-Linear Maps

We can also ask what are the D^{F}-Linear Maps:

$$\mathsf{D}^{\mathsf{F}}[f](x,y) = f(y)$$

Proposition

In a CFC$_k$, D^{F}-Linear $\Leftrightarrow$ k-linear.

There are CDC$_k$ where some k-linear maps are not D-linear maps!

So Fermat theories where being D^{F}-linear and being k-linear are the same are indeed CFC$_k$.

One way to fix this issue would be consider to use **systems of linear maps**.

Blute, R., Cockett, J.R.B. and Seely, R.A., **Cartesian differential storage categories**.

Cockett, R., Lemay, J-S.P., **Linearizing Combinators**.

Which almost works: I can prove 6/7 axioms.... I'm struck on the analogue of **[CD.7]**...

More Remarks and Things Left to Look At

- We can also take partial versions of $\mathsf{F}[-]$, that is, for every map $f : C \times A \to B$, we can build a unique map $\mathsf{F}^C[f] : C \times A \times A \times A \to B$ which satisfies context versions of **[F.1]** and **[F.2]**

$$\mathsf{F}^C[f](c, x, y, z) = \mathsf{F}[f]\left((c, x), (0, y), (0, z)\right)$$

  In other words, for a CFC_k $\mathbb{X}$, every simple slice category $\mathbb{X}[C]$ is a CFC_k.

- From any CFC_k we can build a Fermat theory using an internal semiring object.

- We should be able to build cofree CFC_k using the higher order chain rule for $\mathsf{F}[-]$.

- The chain rule for $\mathsf{F}[-]$ induces an endofunctor H defined as:

$$\mathsf{H}(A) = A \times A \times A \qquad \mathsf{H}(f)(x, y, z) = \left(f(x), \mathsf{F}[f](x, y, y), \mathsf{F}[f](x, y, z)\right)$$

  Is this possibly a tangent category? Or what is the Fermat analogue of a tangent category?

- If a CDC_k has integration, can it be a CFC_k?

- There is another axiomatization of Fermat theories using *difference quotients*, that is, for every map $f(x)$, there exists a unique map $\Delta(f)(x, y)$ such that:

$$f(x) - f(y) = \Delta(f)(x, y)(x - y)$$

  Note that $\Delta(f)$ is not the same as $\tilde{f}$ but we can build one from the other. In this case the derivative is $f'(x) = \Delta(f)(x, x)$. Carlos and I have also given the $\mathsf{CD}/\mathsf{FC}_k$ version of this.

Carchedi, D. & Roytenberg, D. **On Theories of Superalgebras of Differentiable Functions**.

Thanks for listening! Merci!

Paper in progress (maybe?).

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