CAP – a categorical (re)organization of computer algebra

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Linear PDE system with polynomial coefficients

Motivating application: Compute the space

$$\mathrm{Sol}(\Delta) \coloneqq \left\{ \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \end{pmatrix} \mid f, g \in C^\infty(\mathbb{R}^3, \mathbb{R}) \right\}$$

of smooth solutions of the linear PDE system

$$\underbrace{\begin{array}{rr} (\partial_y\partial_z - \frac{1}{3}\partial_z^2 + \frac{1}{3}\partial_x + \partial_y - \frac{1}{3}\partial_z) f \;+ & (\partial_y\partial_z - \frac{1}{3}\partial_z^2) g = 0 \\ (\partial_x\partial_z + \partial_z^2 + \partial_z) f \;+ & (\partial_x\partial_z + \partial_z^2) g = 0 \\ (\partial_z^2 - \partial_x + \partial_z) f \;+ & (3\partial_x\partial_y + \partial_z^2) g = 0 \\ \partial_x\partial_y f \quad & = 0 \\ (\partial_z^2 - \partial_x + \partial_z) f \;+ & (-3\partial_x^2 + \partial_z^2) g = 0 \\ \partial_x^2 f \quad & = 0 \\ \left(x\partial_z^2 - \left(x - \frac{3}{2}\right)\partial_x + \left(x + \frac{3}{2}\right)\partial_z + \frac{3}{2}\right) f \;+ & (x\partial_z^2 + \frac{3}{2}\partial_x + \frac{3}{2}\partial_z) g = 0 \\ (\partial_z^3 + 2\partial_z^2 + \partial_z) f \;+ & (\partial_z^3 + \partial_x\partial_z + \partial_z^2) g = 0 \end{array}}_{\Delta(f,g) = 0}$$

The algebraic approach – Step 1

Weyl **algebra** D	$D \coloneqq \mathbb{R}[x,y,z]\langle \partial_x, \partial_y, \partial_z \rangle$
matrix $\mathtt{m}_\triangle$ over D	$\mathtt{m}_\triangle \in D^{8\times 2}$
D-**module** $\mathcal{F}$ of smooth functions	$\mathcal{F} \coloneqq C^\infty(\mathbb{R}^3, \mathbb{R})$

$$\underbrace{\underset{\mathtt{m}_\triangle}{\overset{D^{8\times 2}}{\Psi}}}_{\begin{pmatrix} \partial_y\partial_z - \frac{1}{3}\partial_z^2 + \frac{1}{3}\partial_x + \partial_y - \frac{1}{3}\partial_z & \partial_y\partial_z - \frac{1}{3}\partial_z^2 \\ \partial_x\partial_z + \partial_z^2 + \partial_z & \partial_x\partial_z + \partial_z^2 \\ \partial_z^2 - \partial_x + \partial_z & 3\partial_x\partial_y + \partial_z^2 \\ \partial_x\partial_y & 0 \\ \partial_z^2 - \partial_x + \partial_z & -3\partial_x^2 + \partial_z^2 \\ \partial_x^2 & 0 \\ x\partial_z^2 - (x - \frac{3}{2})\partial_x + (x + \frac{3}{2})\partial_z + \frac{3}{2} & x\partial_z^2 + \frac{3}{2}\partial_x + \frac{3}{2}\partial_z \\ \partial_z^3 + 2\partial_z^2 + \partial_z & \partial_z^3 + \partial_x\partial_z + \partial_z^2 \end{pmatrix}} \cdot \underbrace{\underset{\psi}{\overset{\mathcal{F}^2}{\Psi}}}_{\begin{pmatrix} f \\ g \end{pmatrix}} = \underbrace{\underset{0}{\overset{\mathcal{F}^8}{\Psi}}}_{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$$

The algebraic approach – Step 2

- $D := \mathbb{R}[x_1, \ldots, x_n]\langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle$, $\mathtt{m}_\Delta \in D^{p \times q}$, $\mathcal{F} = C^\infty(\mathbb{R}^n, \mathbb{R})$
- $\Delta : \mathtt{m}_\Delta \cdot \psi = 0, \quad \psi \in \mathcal{F}^q$.

Interpret the matrix $\mathtt{m}_\Delta$ as a morphism of free D-modules:

Definition

Define the D-**module** M_Δ as the f.p. D-module

$$M_\Delta := D^{1\times q} / \operatorname{im}\left(D^{1\times p} \xrightarrow{\mathtt{m}_\Delta} D^{1\times q} \right) = D^{1\times q} / \left(D^{1\times p} \cdot \mathtt{m}_\Delta \right)$$
$$=: \operatorname{coker}\left(D^{1\times p} \xrightarrow{\mathtt{m}_\Delta} D^{1\times q} \right).$$

The residue classes $(\overline{e}_1, \ldots, \overline{e}_q)$ of the standard basis of the free D-module $D^{1\times q}$ is a generating system of M_Δ.

The rows of $\mathtt{m}_\Delta$ are the defining relations between $\overline{e}_1, \ldots, \overline{e}_q$:

$$\mathtt{m}_\Delta \cdot \begin{pmatrix} \overline{e}_1 \\ \vdots \\ \overline{e}_q \end{pmatrix} = 0.$$

The algebraic approach – Step 3

We therefore call $\begin{pmatrix} \overline{e}_1 \\ \vdots \\ \overline{e}_q \end{pmatrix}$ the abstract solution of $\mathtt{m}_\Delta \psi = 0$.

Lemma von Noether-Malgrange

The map

$$\begin{array}{ccc} \mathrm{Hom}(M_\Delta, \mathcal{F}) & \xrightarrow{\sim} & \mathrm{Sol}(\Delta, \mathcal{F}) \\ \varphi := (\overline{e}_i \mapsto f_i) & \mapsto & \psi := (f_i) \in \mathcal{F}^q \end{array}$$

is an isomorphism of $\mathbb{R}$-vector spaces.

The lemma implies that:

- $\mathrm{Sol}(\Delta, \mathcal{F})$ only depends on the isomorphism type of M_Δ.
- The D-module M_Δ can be studied *independent* of $\mathcal{F}$.
- A different generating set of M_Δ yields an equivalent system Δ' of linear PDEs with $M_\Delta \cong M_{\Delta'}$.

The algebraic approach – Step 4

Given a finitely presented D-module M:

The bidualizing spectral sequence

$$E^2_{pq} = \mathrm{Ext}^{-p}(\mathrm{Ext}^q(M,D),D)) \Longrightarrow M \quad \text{for } p+q=0$$

gives rise to the so-called **purity filtration** of M.

We can use this filtration to solve the above linear PDE system.

Software demo

The `homalg` project

Computing spectral sequences and their induced filtrations required computational models for:

- The abelian category D-$\mathbf{fpmod}$ of f.p. D-modules
- Diagram chasing in abelian categories

Both were realized in the `homalg` project:

- D-$\mathbf{fpmod}$ was implemented as an abelian category
- The only modular part of the implementation was D
- Depending on D, the implementation required various NF-algorithms up to noncommutative Gröbner bases
- Diagram chasing was realized by **generalized morphisms**

The motivation for the `CAP` project

- `homalg` was well-desigend for the intended application
- however, not modular enough to cover more applications
- implementing more complicated categories became increasingly difficult, e.g.,
- generalizing from f.p. modules to coh. sheaves was a pain

Rectify: Take category theory more seriously

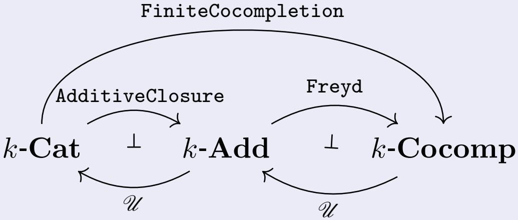
- category theory should guide all design decisions
- categories, functors, ... should become first class citizens
- turn category theory into a programming language:
- write all algorithms using categorical vocabulary

Revisiting D-**fpmod**

What is D-**fpmod** categorically?

- View D as a k-linear category on one object
- D-**fpmod** is the *finite colimit completion* of D

`FiniteCocompletion` as a **categorical tower** of biadjunctions



- `AdditiveClosure` formally adds direct sums
- `AdditiveClosure` invents matrices
- `Freyd` formally adds cokernels
- `Freyd` is a quotient of the arrow category

Free-forgetful 2-adjunctions

The above tower of categorical constructors is typically composed of several free-forgetful 2-adjunctions

$$\mathcal{D} \underset{\mathscr{U}}{\overset{\mathscr{L}}{\underset{\bot}{\rightleftarrows}}} \mathcal{E}$$

between a 2-category $\mathcal{D}$ of categories (called **doctrine**) and another doctrine $\mathcal{E}$ of categories with extra structure.

Software demo

FiniteCompletions

The dual category construction is also a 2-adjunction on each doctrine

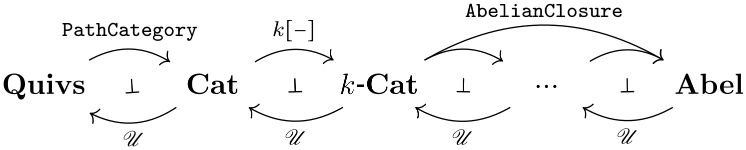
$$\mathcal{D} \underset{\mathscr{R} = \texttt{Opposite}}{\overset{\mathscr{L} = \texttt{Opposite}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{D}^{\mathrm{co-dual}}$$

Implementing `Opposite` requires a lot of meta programming.

More categorical towers of biadjunctions

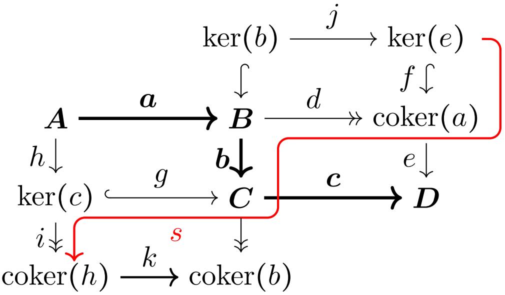
- $\texttt{CoFreyd} \coloneqq \texttt{Opposite} \circ \texttt{Freyd} \circ \texttt{Opposite}$
- $\texttt{FiniteCompletion} \coloneqq \texttt{Opposite} \circ \texttt{FiniteCocompletion} \circ \texttt{Opposite}$
- $\texttt{FpCoPreSheaves} \coloneqq \texttt{Opposite} \circ \texttt{FpPreSheaves} \circ \texttt{Opposite}$

A longer categorical tower of biadjunction yields `AbelianClosure` as a **categorical tower** of 2-*adjunctions*:



Simplest diagram chasing: The connecting morphism

Snake Lemma: Given three composable morphisms $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$ in an Abelian category with $abc = 0$.



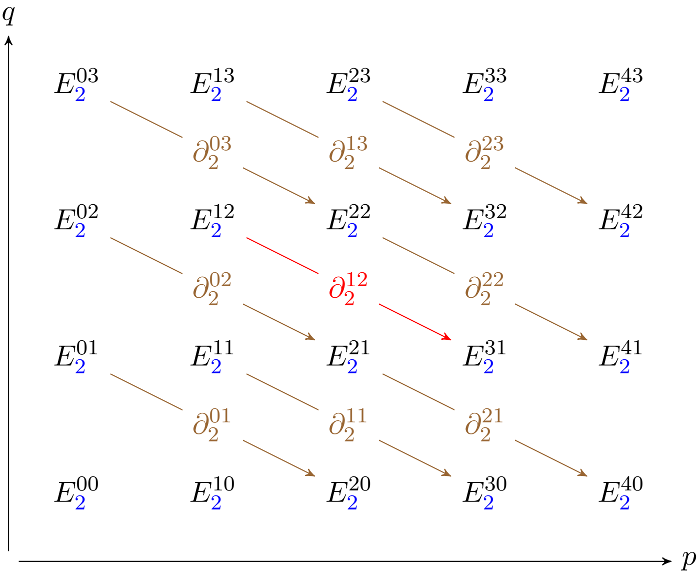
$\rightsquigarrow$ $\exists$ an *ess. unique natural* morphism $\ker(e) \xrightarrow{s} \operatorname{coker}(h)$ with $\ker(b) \xrightarrow{j} \ker(e) \xrightarrow{s} \operatorname{coker}(h) \xrightarrow{k} \operatorname{coker}(b)$ an exact sequence.

Software demo

`https://homalg-project.github.io/nb/SnakeInFreeAbelian`

Exercise: Along the same lines treat spectral sequences of bicomplexes.

Spectral sequences of bicomplexes



Examples of categorical towers

We can model

- free left R-modules of finite rank via $\mathcal{C}(R)^{\oplus}$
- free right R-modules of finite rank via $(\mathcal{C}(R)^{\oplus})^{\mathrm{op}}$
- finitely presented left R-modules via $\mathbf{Freyd}(\mathcal{C}(R)^{\oplus})$
- finitely presented right R-modules via $\mathbf{Freyd}((\mathcal{C}(R)^{\oplus})^{\mathrm{op}})$
- quivers via $\mathbf{Func}(\mathcal{C}(\mathfrak{A} \rightrightarrows \mathfrak{V}), \mathbf{Sets})$
- ZX-diagrams via $\mathbf{Sub}(\mathbf{Csp}(\mathbf{Slice}(\mathbf{Func}(\mathcal{C}(\mathfrak{A} \rightrightarrows \mathfrak{V}), \mathbf{Sets}))))$
- free Abelian categories for theorem proving via $\mathbf{Freyd}(\mathbf{Freyd}(-^{\mathrm{op}})^{\mathrm{op}})$
- linear representations of a group G over a field k via $\mathbf{Func}(\mathcal{C}(G), k^{\oplus})$
- radical ideals of a ring R via $\mathbf{StablePoset}(\mathbf{Poset}(\mathbf{Slice}(\mathcal{C}(R)^{\oplus})))$

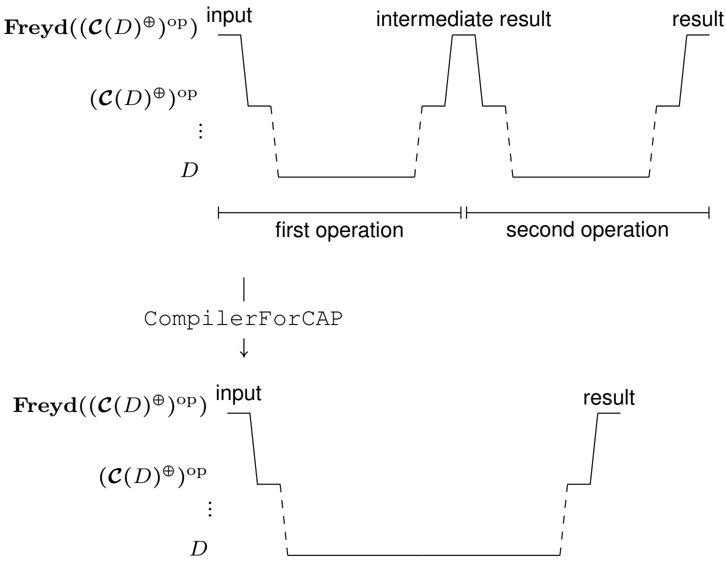
Advantages:

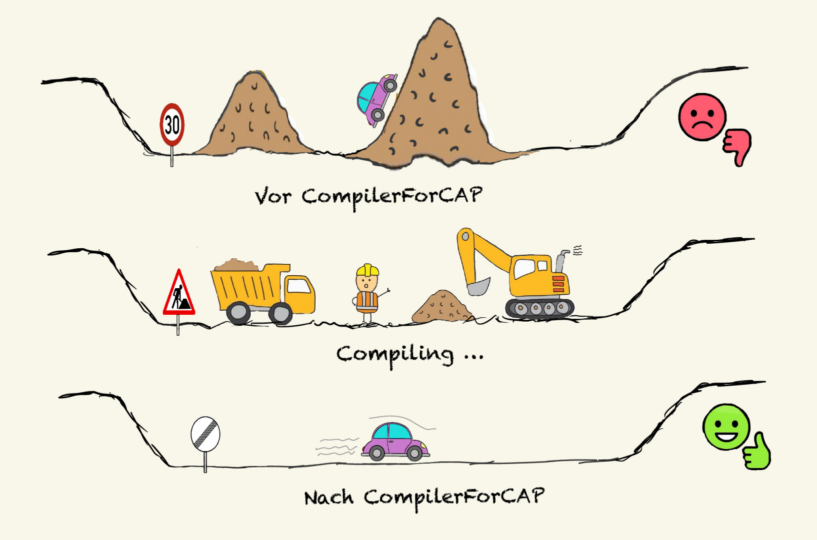
- **Reusability**: Building blocks can appear in multiple different contexts.
- **Separation of concerns**: Focus on a single concept at a time.
- **Verifiability**: Every constructor has a limited scope.
- **Emergence**: The whole is greater than the sum of its parts.

Effects on computer implementations

- **Efficient development** thanks to
  - reusability
  - separation of concerns
  - verifiability
  - emergence
- **Inefficient execution** due to computational overhead :-(
- Solution: compilation

Overhead of boxing and unboxing




30
Vor CompilerForCAP
Compiling ...
Nach CompilerForCAP

Benchmarks

Consider a computation in the categorical tower

$$\mathbf{Freyd}((\mathcal{C}(D)^{\oplus})^{\mathrm{op}}) \simeq \textbf{fpmod}\text{-}D$$

problem size	**original code** (s)	**compiled code** (s)	**factor**
1	0.2	0.05	≈ 5
2	2.4	0.06	≈ 50
3	19.1	0.07	≈ 250
4	118.9	0.09	≈ 1250
5	584.5	0.12	≈ 5000
10	N/A	0.35	N/A
20	N/A	1.34	N/A
30	N/A	3.53	N/A

We see a difference between “finishes in seconds” and “will never finish”.

Further applications

`CompilerForCAP` can also be used

- for removing additional sources of overhead,
- for **generating categorical code** from categorical towers,
- as a **proof assistant** for verifying categorical implementations.

Conclusion

- **Algorithmic category theory** is a high-level programming language.
- Using this language for building **categorical towers** allows
  - to reach a wide range of advanced and complex applications
  - allowing reusability, separation of concerns, verifiability, and emergence.
- This approach naturally comes with a computational overhead.
- `CompilerForCAP` can avoid this overhead, allowing us to make full use of the advantages of building categorical towers.

Thank you