Double Categorical Equivalences

joint with Lyne Moser and Paula Verdugo

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EX:

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equivalences		

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Recall...

A 2-category has...

objects $\bullet \quad \bullet \quad \bullet$

morphisms $\bullet \longrightarrow \bullet$

2-cells
$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

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$$\begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd}$$

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$$\begin{tikzcd} \bullet \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, bend right=50, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$

A double category has...

objects $\bullet \quad \bullet \quad \bullet$

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vertical morphisms
$$\begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd}$$

squares
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

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Recall... for 2-categories:

Defn A 2-functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a biequivalence if it's

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$$\begin{tikzcd} FA \arrow[r, bend left, "Ff"{name=U}] \arrow[r, bend right, "g"'{name=D}] & FA' \arrow[from=U, to=D, phantom, "\Downarrow \cong"] \end{tikzcd}$$

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horizontal equivalence

$$\begin{tikzcd} FA \arrow[r] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \cong"] & B \arrow[r] & FA \arrow[d, equal] \\ FA \arrow[rr, equal] & & FA \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \cong"] & FA \arrow[r] & B \arrow[d, equal] \\ B \arrow[rr, equal] & & B \end{tikzcd}$$

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- Looking at $\mathrm{DblCat} = \mathrm{Cat}(\mathrm{Cat})$

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  $$F_0 : \mathbb{A}_0 \longrightarrow \mathbb{B}_0 , \qquad F_1 : \mathbb{A}_1 \longrightarrow \mathbb{B}_1$$

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Apply to $\mathcal{A} = \mathrm{DblCat}_h$ (double cats / double functors / hor. nat. tr)

$F : \mathbb{A} \to \mathbb{B}$ weak equiv $\iff \exists\, G : \mathbb{B} \to \mathbb{A}$ + hor. nat. isos

$$FG \cong \mathrm{id}, \quad GF \cong \mathrm{id}.$$

- Recall: $T : \mathcal{A} \longrightarrow \mathcal{A}$ 2-monad $\rightsquigarrow$ there is a model str. on T-algebras with weak equivs = underlying morphism is an equiv. in $\mathcal{A}$ [Lack].

- Recall: $T : A \longrightarrow A$ 2-monad $\rightsquigarrow$ there is a model str. on T-algebras with weak equivs = underlying morphism is an equiv. in A [Lack].

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All these studied by Fiore–Paoli–Pronk [FPP].

Super interesting — but not quite what we're looking for

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HOW? Using homotopy theory.

ASSUMPTION 1 Any good notion of equivalence will be part of a model structure on DblCat.

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UPSHOT: now there is a lot more structure we can use.

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equivalences: • surj. on obj. up to iso, $FC \xrightarrow{\cong} D$ • fully faithful.	**biequivalences:** • surj. on obj. up to equiv, $FC \xrightarrow{\simeq} D$ • full on mor • fully faithful on 2-cells	?
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ASSUMPTION 1 Any good notion of equivalence will be part of a model structure.

ASSUMPTION 2 These should be the "canonical trivial fibrations",

i.e. the canonical equivalences should be weak equivalences in a model structure with this class of trivial fibrations.

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$$\emptyset \xrightarrow{i_1} \bullet \qquad\qquad \bullet \quad \bullet \xrightarrow{i_2} \bullet \longrightarrow \bullet \qquad\qquad \bullet \quad \bullet \xrightarrow{i_3} \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}$$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \xrightarrow{i_4} \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\ \Downarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \xrightarrow{i_5} \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

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<u>Thm</u> [MSV] Every model structure on DblCat with the canonical trivial fibrations is left proper.

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trivial fibrations: • surj. on obj. i.e. $FC = D$ • fully faithful	trivial fibrations: • surj. on obj. • full on mor • fully faithful on 2-cells	trivial fibrations: • surj. on obj. • full on hor & ver mor • fully faithful on squares

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Defn A companion pair in a double category $\mathbb{A}$ is the data of

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\varphi"] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\psi"] & A \arrow[d, "u"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

such that

$$
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A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\psi"] & A \arrow[d, "u"] \\
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Defn A gregarious equivalence in $\mathbb{A}$ is a companion pair (f, u, φ, ψ) such that:

- f is a horizontal equiv
- u is a vertical equiv.

Defn [Campbell] $F : \mathbb{A} \to \mathbb{B}$ is a gregarious double equiv. if it's:

- surjective on obj. up to gregarious equiv.

  i.e. both
$$\begin{tikzcd} FA \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & B \\ B & \end{tikzcd}$$
+ compatibility

- full on horizontal & vertical mor. up to globular iso

  i.e.
$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & FA' \arrow[d, equal] \\ FA \arrow[r, "g"'] & FA' \end{tikzcd} \quad , \quad \begin{tikzcd} FA \arrow[r, equal] \arrow[d, "Fu"'] \arrow[dr, phantom, "\cong"] & FA \arrow[d, "v"] \\ FA' \arrow[r, equal] & FA' \end{tikzcd}$$

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Thm [Campbell, MSV] There is a model str. on DblCat with:

- weak equivalences = gregarious double equivs
- trivial fibrations = canonical trivial fibrations

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In other words... the gregarious double equivalences are the minimal class of equivalences compatible w/ the canonical trivial fibrations, and any others are obtained by localizing these.

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In other other words... any other such class of equivalences contains the gregarious ones.

Not the only evidence in favor of gregarious:

[Campbell] The gregarious model structure is the model structure on DblCat w/ the canonical trivial fibrations in which all objects are fibrant.

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[Leinster] $\mathbb{A}, \mathbb{B} \in$ DblCat are gregariously double equivalent $\iff$ there exists a span $\mathbb{A} \leftarrow \mathbb{C} \longrightarrow \mathbb{B}$ of canonical trivial fibrations.

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[Leinster] $\mathbb{A}, \mathbb{B} \in$ DblCat are gregariously double equivalent $\Leftrightarrow$ there exists a span $\mathbb{A} \leftarrow \mathbb{C} \rightarrow \mathbb{B}$ of canonical trivial fibrations.

Both of these mirror the corresponding scenario in Cat, 2Cat.

FIRST GOAL: ✓

NEW GOAL: Further understand / construct ex's of other model structures w/ canonical trivial fibrations.

Thm [MSV] Any (combinatorial) model structure on DblCat w/ the canonical trivial fibrations is a (Bousfield) localization of the gregarious model structure.

In theory, this gives a complete answer to GOAL 2.

In practice, Bousfield localizations can be tricky to understand.

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FACT

$$\left.\begin{array}{l}\cdot\ \text{trivial fibrations}\\ \cdot\ \text{fibrant objects}\end{array}\right\}\ \text{completely determine the model structure}$$

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Thm [MSV] Any combinatorial model str. on DblCat w/ the canonical trivial fibrations arises from this recipe.

EXAMPLES:

	fibrant objects	properties
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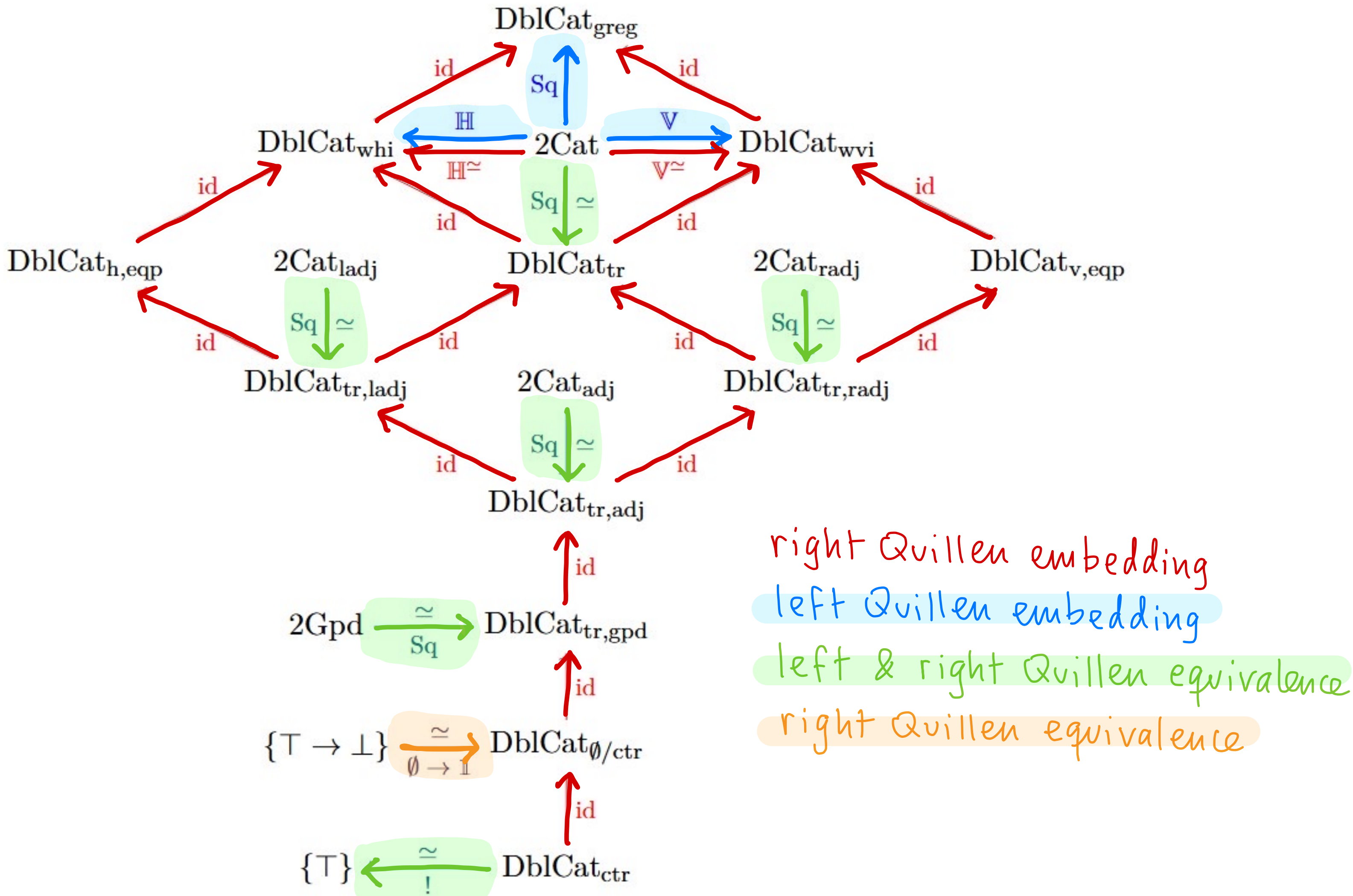
$\mathrm{DblCat}_{tr,gpd}$	double groupoids + every hor. & ver. mor. has a companion	homotopy theory of 2-groupoids


DblCat_greg
2Cat
DblCat_whi
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DblCat_h,eqp
2Cat_ladj
DblCat_tr
2Cat_radj
DblCat_v,eqp
DblCat_tr,ladj
2Cat_adj
DblCat_tr,radj
DblCat_tr,adj
2Gpd
DblCat_tr,gpd
{⊤ → ⊥}
DblCat_∅/ctr
{⊤}
DblCat_ctr
id
Sq
ℍ
𝕍
ℍ^≃
𝕍^≃
≃
∅ → 𝟙
!
right Quillen embedding
left Quillen embedding
left & right Quillen equivalence
right Quillen equivalence

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