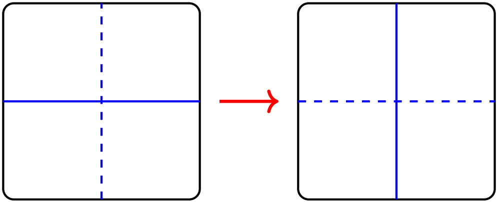
The Eckmann-Hilton argument in duoidal categories



Marcelo Aguiar, Cornell University
Javier Cóppola, Universidad de la República de Uruguay

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DUOIDAL CATEGORIES

The Eckmann-Hilton argument

Consider a set with two binary operations $+$ and $\bullet$ and two elements 0 and 1 such that

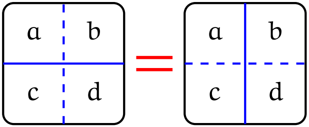
$$(a + b) \bullet (c + d) = (a \bullet c) + (b \bullet d),$$
$$a + 0 = a = 0 + a,$$
$$a \bullet 1 = a = 1 \bullet a.$$

Then:

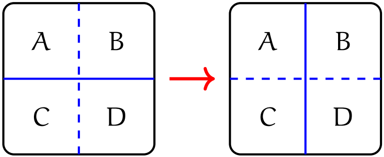
$$0 = 1, \quad + = \bullet,$$

and this operation is both commutative and associative.

Eckmann-Hilton axiom:



A duoidal category is a context in which these equalities are replaced by coherent maps.



Duoidal categories

Let $\mathscr{C}$ be a category. A duoidal structure on $\mathscr{C}$ consists of:

- Two monoidal structures on $\mathscr{C}$: two tensor products

$$\mathscr{C} \times \mathscr{C} \xrightarrow{\diamond} \mathscr{C} \quad \text{and} \quad \mathscr{C} \times \mathscr{C} \xrightarrow{\star} \mathscr{C}$$

  with respective unit objects J and K.

- A natural transformation

$$\zeta_{A,B,C,D} \colon (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$

  called the interchange law.

- Three morphisms

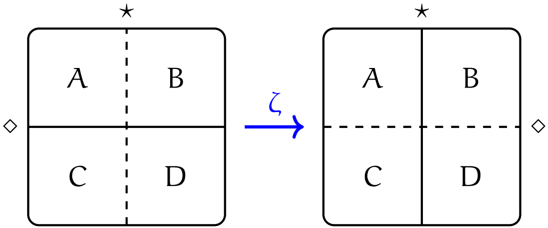
$$\delta_0 \colon J \to J \star J, \qquad \mu_0 \colon K \diamond K \to K, \qquad \zeta_0 \colon J \to K.$$

All of these are subject to various axioms.

Complete definition by Garner and (independently) A.-Mahajan.
Precedents in work of Balteanu-Fiedorowicz and others.

The interchange law

$$(A \star B) \diamond (C \star D) \xrightarrow{\zeta} (A \diamond C) \star (B \diamond D)$$



- The interchange law is not required to be invertible.
- The order of the operations $(\diamond, \star)$ matters.

Axioms: $\star$ is monoidal with respect to $\diamond$,
$\diamond$ is comonoidal with respect to $\star$.

Duoidal categories: the axioms

Recall the structure maps:

$$(A \star B) \diamond (C \star D) \xrightarrow{\zeta_{A,B,C,D}} (A \diamond C) \star (B \diamond D),$$

$$J \xrightarrow{\zeta_J} J \star J, \qquad J \xrightarrow{\zeta_0} K, \qquad K \diamond K \xrightarrow{\zeta_K} K.$$

Let $\mathscr{C} \times \mathscr{C} \underset{\mathscr{G}}{\overset{\mathscr{F}}{\rightrightarrows}} \mathscr{C}$ be $\mathscr{F}(A, B) = A \star B$ and $\mathscr{G}(A, B) = A \diamond B$.

The axioms are:

- The functor $\mathscr{F}$ is monoidal with respect to $\diamond$ with

$$\mathscr{F}(A, B) \diamond \mathscr{F}(C, D) \xrightarrow{\zeta_{A,B,C,D}} \mathscr{F}\big((A, B) \diamond (C, D)\big) \quad \text{and} \quad J \xrightarrow{\zeta_J} \mathscr{F}(J, J).$$

- (K, ζ_K, ζ_0) is a monoid in $(\mathscr{C}, \diamond, J)$.
- The functor $\mathscr{G}$ is comonoidal with respect to $\star$ with

$$\mathscr{G}\big((A, C) \star (B, D)\big) \xrightarrow{\zeta_{A,B,C,D}} \mathscr{G}(A, C) \diamond \mathscr{G}(B, D) \quad \text{and} \quad \mathscr{G}(K, K) \xrightarrow{\zeta_K} K.$$

- (J, ζ_J, ζ_0) is a comonoid in $(\mathscr{C}, \star, K)$.

Duoidal categories: two of the axioms

$$\begin{array}{ccc} & (A \star X) \diamond (B \star Y) \diamond (C \star Z) & \\ \swarrow^{\zeta \diamond \mathrm{id}} & & \searrow^{\mathrm{id} \diamond \zeta} \\ \big((A \diamond B) \star (X \diamond Y)\big) \diamond (C \star Z) & & (A \star X) \diamond \big((B \diamond C) \star (Y \diamond Z)\big) \\ \searrow^{\zeta} & & \swarrow^{\zeta} \\ & (A \diamond B \diamond C) \star (X \diamond Y \diamond Z) & \end{array}$$

$$\begin{array}{ccc} & (A \star B \star C) \diamond (X \star Y \star Z) & \\ \swarrow^{\zeta} & & \searrow^{\zeta} \\ \big((A \star B) \diamond (X \star Y)\big) \star (C \diamond Z) & & (A \diamond X) \star \big((B \star C) \diamond (Y \star Z)\big) \\ \searrow^{\zeta \star \mathrm{id}} & & \swarrow^{\mathrm{id} \star \zeta} \\ & (A \diamond X) \star (B \diamond Y) \star (C \diamond Z) & \end{array}$$

The remaining axioms involve the unit objects and unit maps.

Example: braided monoidal categories

Let $(\mathscr{C}, \otimes, \mathrm{I}, \beta)$ be a braided monoidal category with braiding

$$\beta_{A,B} : A \otimes B \to B \otimes A.$$

Then $\mathscr{C}$ is duoidal with

$$\diamond = \star = \otimes \qquad \text{and} \qquad J = K = I.$$

The interchange law is

$$A \otimes B \otimes C \otimes D \xrightarrow{\mathrm{id} \otimes \beta \otimes \mathrm{id}} A \otimes C \otimes B \otimes D.$$

Proposition (Joyal-Street).
Let $\mathscr{C}$ be a duoidal category in which all structure maps are invertible. Then the duoidal structure arises from a braided monoidal structure as above.

Example: M-graded modules

Let M be a monoid and $\Bbbk$ a commutative ring.
Let $\mathscr{C}$ be the category of M-graded $\Bbbk$-modules:

$$X = (X_m)_{m \in M}, \text{ each } X_m \text{ is a } \Bbbk\text{-module.}$$

Then $\mathscr{C}$ is duoidal with $\diamond$ and $\star$ defined by

$$(X \diamond Y)_m = \bigoplus_{p \cdot q = m} X_p \otimes Y_q \qquad \text{and} \qquad (X \star Y)_m = X_m \otimes Y_m.$$

The unit objects J and K are defined by

$$J_m = \begin{cases} \Bbbk & \text{if } m = 1 \\ 0 & \text{if not} \end{cases} \qquad \text{and} \qquad K_m = \Bbbk \text{ for all } m.$$

Example: spans

Let X be a fixed set. Let $\mathscr{C}$ be the category of spans over X.

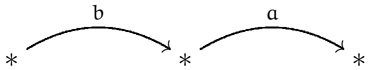
The objects are triples (A, s, t) where $A \overset{s}{\underset{t}{\rightrightarrows}} X$.

Spans are also called digraphs or quivers with vertex set X:

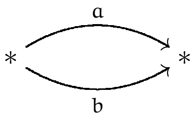
$$s(a) \xrightarrow{\ a\ } t(a)\ .$$

Then $\mathscr{C}$ is duoidal with $\diamond$ and $\star$ defined by

$$A \diamond B = \{(a, b) \in A \times B : s(a) = t(b)\},$$
$$A \star B = \{(a, b) \in A \times B : s(a) = s(b) \text{ and } t(a) = t(b)\}.$$



in series



in parallel

Example

Let $(\mathscr{C}, \otimes)$ be an arbitrary monoidal category.

Suppose that in $\mathscr{C}$ all finite products exist and consider the corresponding cartesian monoidal category $(\mathscr{C}, \times)$.

Then $(\mathscr{C}, \otimes, \times)$ is a duoidal category: the interchange law

$$\zeta_{A,B,C,D} \colon (A \times B) \otimes (C \times D) \to (A \otimes C) \times (B \otimes D)$$

is the unique map with components

$$\pi_A^{A \times B} \otimes \pi_C^{C \times D} \quad \text{and} \quad \pi_B^{A \times B} \otimes \pi_D^{C \times D},$$

where π denotes the canonical projections.

The duoidal category of spans is of this form.

Example: bimodules

Let R be a commutative $\Bbbk$-algebra.
Let $\mathscr{C}$ be the category of R-bimodules, or equivalently, of $R \otimes R$-modules.

Then $\mathscr{C}$ is duoidal with $\diamond$ and $\star$ defined by

$$X \diamond Y = X \otimes_{R \otimes R} Y \qquad \text{and} \qquad X \star Y = X \otimes_R Y.$$

The unit objects are $J = R \otimes R$ and $K = R$.

The interchange law is induced from the braiding on $\Bbbk$-modules:

$$\begin{array}{ccc}
(A \otimes B) \otimes (C \otimes D) & \xrightarrow{\mathrm{id} \otimes \beta \otimes \mathrm{id}} & (A \otimes C) \otimes (B \otimes D) \\
\downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\
(A \star B) \otimes (C \star D) & & (A \diamond C) \otimes (B \diamond D) \\
\downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\
(A \star B) \diamond (C \star D) & \underset{\zeta}{\dashrightarrow} & (A \diamond C) \star (B \diamond D)
\end{array}$$

ALGEBRAIC STRUCTURES IN DUOIDAL CATEGORIES

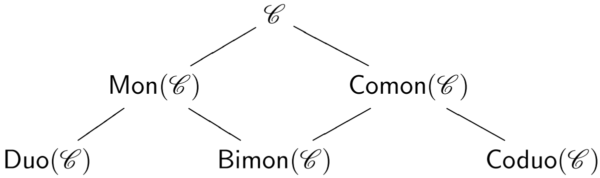
Bimonoids, duoids, coduoids

Proposition. Let $(\mathscr{C}, \diamond, \star, \zeta)$ be a duoidal category. Then $\mathrm{Mon}(\mathscr{C}, \diamond)$ is monoidal under $\star$ and $\mathrm{Comon}(\mathscr{C}, \star)$ is monoidal under $\diamond$. In addition,

$$\mathrm{Comon}(\mathrm{Mon}(\mathscr{C}, \diamond), \star) = \mathrm{Mon}(\mathrm{Comon}(\mathscr{C}, \star), \diamond).$$

Define bimonoids, duoids, and coduoids in $\mathscr{C}$ by

- $\mathrm{Bimon}(\mathscr{C}, \diamond, \star) = \mathrm{Comon}(\mathrm{Mon}(\mathscr{C}, \diamond), \star) = \mathrm{Mon}(\mathrm{Comon}(\mathscr{C}, \star), \diamond)$,
- $\mathrm{Duo}(\mathscr{C}, \diamond, \star) = \mathrm{Mon}(\mathrm{Mon}(\mathscr{C}, \diamond), \star)$,
- $\mathrm{Coduo}(\mathscr{C}, \diamond, \star) = \mathrm{Comon}(\mathrm{Comon}(\mathscr{C}, \star), \diamond)$.



Bimonoids

A bimonoid B in a duoidal category $\mathscr{C}$ has

$$B \diamond B \xrightarrow{\mu} B \quad \text{and} \quad B \xrightarrow{\delta} B \star B$$

subject (among other axioms) to

$$\begin{array}{ccc}
(B \star B) \diamond (B \star B) & \xrightarrow{\zeta} & (B \diamond B) \star (B \diamond B) \\
\uparrow{\scriptstyle \delta \diamond \delta} & & \downarrow{\scriptstyle \mu \star \mu} \\
B \diamond B \xrightarrow{\mu} & B & \xrightarrow{\delta} B \star B.
\end{array}$$

This is equivalent to either of:

- $\mu : B \diamond B \to B$ is a morphism of $\star$-comonoids,
- $\delta : B \to B \star B$ is a morphism of $\diamond$-monoids.

Bimonoids: examples

- In a braided monoidal category, the notion of bimonoid acquires its usual meaning:

$$B \otimes B \xrightarrow{\mu} B \quad \text{and} \quad B \xrightarrow{\delta} B \otimes B.$$

- In $(\mathscr{C}, \otimes, \times)$, we have

$$\mathsf{Bimon}(\mathscr{C}, \otimes, \times) = \mathsf{Mon}(\mathscr{C}, \otimes).$$

- In particular, a bimonoid in spans over X is a category with object set X.
- Replacing spans for X-bigraded $\Bbbk$-modules, one obtains a duoidal category in which bimonoids are the semi-Hopf categories of Batista, Caenepeel, and Vercruysse.
- A bimonoid B in M-graded modules is a semi-Hopf M-algebra in the sense of Turaev. It carries operations

$$B_p \otimes B_q \xrightarrow{\mu_{p,q}} B_{p \cdot q} \quad \text{and} \quad B_m \xrightarrow{\Delta_m} B_m \otimes B_m.$$

- A bimonoid in R-bimodules is a bialgebroid with commutative base R.

Duoids

A duoid D in a duoidal category $\mathscr{C}$ has

$$\mathrm{D} \diamond \mathrm{D} \xrightarrow{\mu} \mathrm{D} \quad \text{and} \quad \mathrm{D} \star \mathrm{D} \xrightarrow{\nu} \mathrm{D}$$

subject (among other axioms) to

$$\begin{array}{ccccc}
(\mathrm{D} \star \mathrm{D}) \diamond (\mathrm{D} \star \mathrm{D}) & & \xrightarrow{\zeta} & & (\mathrm{D} \diamond \mathrm{D}) \star (\mathrm{D} \diamond \mathrm{D}) \\
\downarrow{\scriptstyle \nu \diamond \nu} & & & & \downarrow{\scriptstyle \mu \star \mu} \\
\mathrm{D} \diamond \mathrm{D} & \xrightarrow[\mu]{} & \mathrm{D} & \xleftarrow[\nu]{} & \mathrm{D} \star \mathrm{D}.
\end{array}$$

This is equivalent to:

- $\nu : \mathrm{D} \star \mathrm{D} \to \mathrm{D}$ is a morphism of $\diamond$-monoids.

The notion of coduoid is dual.

Duoids in braided monoidal categories

Proposition (Eckmann-Hilton argument).
Let $\mathscr{C}$ be a braided monoidal category.
Let M be a duoid in the associated duoidal category.
Then the two monoid structures of M coincide and are commutative.

A duoid in a braided monoidal category is a commutative monoid.

The classical Eckmann-Hilton argument is the case $\mathscr{C} = (\mathsf{Set}, \times)$.

Question.
Does this hold (in some form) in more general duoidal categories?
Answer later.

Duoids: examples

- In spans over X, a duoid is a category with object set X enriched in the category of (ordinary) monoids.
- A duoid D in M-graded $\mathbb{k}$-modules is an M-graded $\mathbb{k}$-algebra

$$D_p \otimes D_q \to D_{p\cdot q}, \quad a \otimes b \mapsto a \circ b,$$

for which each component D_m is itself a $\mathbb{k}$-algebra

$$D_m \otimes D_m \to D_m, \quad a \otimes a' \mapsto a \bullet b,$$

in such a way that

$$(a \bullet a') \circ (b \bullet b') = (a \circ b) \bullet (a' \circ b').$$

This is the Eckmann-Hilton axiom, but only when $\deg(a) = \deg(a')$ and $\deg(b) = \deg(b')$.

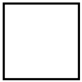
An example when $M = (\mathbb{N}, +)$:
let A be a $\mathbb{k}$-algebra, define $D_n = A^{\otimes n}$, and

$$(a_1 \otimes \cdots \otimes a_p) \circ (b_1 \otimes \cdots \otimes b_q) = a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q,$$
$$(a_1 \otimes \cdots \otimes a_m) \bullet (b_1 \otimes \cdots \otimes b_m) = a_1 b_1 \otimes \cdots \otimes a_m b_m.$$

Then D is a duoid.

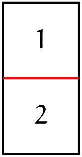
The 2-sphere

Let $S^2 = \mathrm{D}^2/\partial\mathrm{D}^2$. We represent it as a solid square with the boundary identified to a point. This point is the base.

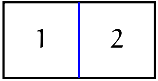


Let $\mathscr{C}$ be the category of based topological spaces with homotopy classes of maps. In this category the coproduct $X \vee Y$ is the join at the base.

We may represent $S^2 \vee S^2$ either as



or



The numbers label the factors in the wedge.

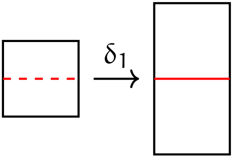
The middle line is identified, along with the boundary, to the base point.

The most famous coduoid

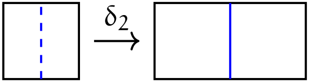
The category $(\mathscr{C}, \vee)$ is symmetric monoidal. (In fact, cocartesian.)
View it as a duoidal category $(\mathscr{C}, \vee, \vee)$.

Proposition. S^2 is a coduoid in $\mathscr{C}$. The coproducts are as follows:

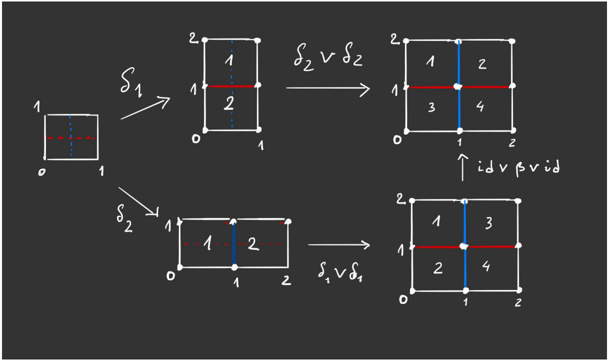
$$\delta_1 : S^2 \to S^2 \vee S^2$$
$$\delta_1(x, y) = (x, 2y)$$



$$\delta_2 : S^2 \to S^2 \vee S^2$$
$$\delta_2(x, y) = (2x, y)$$



The coduoid axiom



Thus, S^2 is a coduoid.
But $\mathscr{C}$ is a symmetric monoidal category.
By Eckmann-Hilton, S^2 is a cocommutative comonoid.

Therefore, $\pi_2(X) = \mathrm{Hom}_{\mathscr{C}}(S^2, X)$ is commutative.

NORMAL DUOIDAL CATEGORIES

Two canonical maps

Let $(\mathscr{C}, \diamond, J, \star, K, \zeta)$ be a duoidal category.

Define (after Garner and López Franco) two transformations:

$$\sigma : X \diamond Y \cong (X \star K) \diamond (K \star Y) \xrightarrow{\zeta} (X \diamond K) \star (K \diamond Y),$$
$$\tau : X \diamond Y \cong (K \star X) \diamond (Y \star K) \xrightarrow{\zeta} (K \diamond Y) \star (X \diamond K).$$

Note: when $(\mathscr{C}, \otimes, \beta)$ is braided monoidal,

$$\sigma = \mathrm{id} : X \otimes Y \to X \otimes Y \qquad \text{and} \qquad \tau = \beta : X \otimes Y \to Y \otimes X.$$

Normal duoidal categories

A duoidal category $\mathscr{C}$ is normal if $J = K$.
(More precisely, if $\zeta_0 : J \to K$ is invertible.)

In this case,

$$\sigma : X \diamond Y \to X \star Y \qquad \text{and} \qquad \tau : X \diamond Y \to Y \star X.$$

Examples of normal duoidal categories.

- Any braided monoidal category.
- The category of pointed sets under coproduct and product:

$$(X, x_0) \bigvee (Y, y_0) = \big((X \sqcup Y)/(x_0 \equiv y_0),\ x_0 \equiv y_0\big),$$
$$(X, x_0) \times (Y, y_0) = \big(X \times Y,\ (x_0, y_0)\big).$$

 The common unit object is a singleton.

The duoidal category of cosimplicial modules

Let $\mathscr{D}$ be the category of cosimplicial $\mathbb{k}$-modules:

$$X = \left(X^n,\, d^i,\, s^j\right), \text{ each } X^n \text{ is a } \mathbb{k}\text{-module.}$$

The diagonal product is

$$(X \times Y)^n = X^n \otimes Y^n.$$

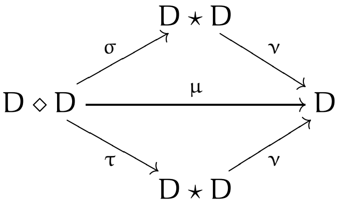
There is another monoidal structure on $\mathscr{D}$ defined by Batanin:

$$(X \diamond Y)^n = \bigoplus_{i+j=n} X^i \otimes Y^j \Big/ \left\langle d^i x \otimes y - x \otimes d^0 y : x \in X^{i-1},\, y \in Y^j \right\rangle.$$

Proposition. $(\mathscr{D}, \diamond, \times)$ is duoidal and normal.
In fact, $J = K =$ the constant cosimplicial module $\mathbb{k}$.

Duoids in normal duoidal categories

Proposition (Garner and López Franco).
Let (D, μ, ν) be a duoid in a normal duoidal category. Then:



This is a generalization of Eckmann-Hilton from the setting of braided monoidal categories to that of normal of duoidal categories.

DUOIDAL FUNCTORS

Functors between duoidal categories

Recall: there are 2 kinds of functors between monoidal categories:

monoidal and comonoidal.

There are 3 kinds of functors between duoidal categories:

bimonoidal, duoidal, and coduoidal.

- Each of these kinds is closed under composition.
- Each kind preserves the corresponding class of objects.
  In particular, duoidal functors preserve duoids.

Duoidal functors

Let $\mathscr{C}$ and $\mathscr{D}$ be two duoidal categories. A (lax) duoidal functor $\mathscr{C} \to \mathscr{D}$ consists of a functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ together with two (lax) monoidal structures

$$\varphi : \mathscr{F}X \diamond \mathscr{F}Y \to \mathscr{F}(X \diamond Y) \qquad \text{and} \qquad \gamma : \mathscr{F}X \star \mathscr{F}Y \to \mathscr{F}(X \star Y).$$

These are subject to various axioms, including:

$$\begin{array}{ccc}
(\mathscr{F}A \star \mathscr{F}B) \diamond (\mathscr{F}C \star \mathscr{F}D) & \xrightarrow{\zeta} & (\mathscr{F}A \diamond \mathscr{F}C) \star (\mathscr{F}B \diamond \mathscr{F}D) \\
\Big\downarrow {\scriptstyle \gamma_{A,B} \diamond \gamma_{C,D}} & & \Big\downarrow {\scriptstyle \varphi_{A,C} \star \varphi_{B,D}} \\
\mathscr{F}(A \star B) \diamond \mathscr{F}(C \star D) & & \mathscr{F}(A \diamond C) \star \mathscr{F}(B \diamond D) \\
\Big\downarrow {\scriptstyle \varphi_{A \star B, C \star D}} & & \Big\downarrow {\scriptstyle \gamma_{A \diamond C, B \diamond D}} \\
\mathscr{F}\big((A \star B) \diamond (C \star D)\big) & \xrightarrow[\mathscr{F}(\zeta)]{} & \mathscr{F}\big((A \diamond C) \star (B \diamond D)\big)
\end{array}$$

Eckmann-Hilton for duoidal functors

Proposition (A.-Cóppola).
Let $\mathscr{C}$ and $\mathscr{D}$ be normal duoidal categories.
Let $(\mathscr{F}, \varphi, \gamma) : \mathscr{C} \to \mathscr{D}$ be a duoidal functor. Then:

$$
\begin{array}{ccc}
\mathscr{F}X \diamond \mathscr{F}Y & \xrightarrow{\sigma} & \mathscr{F}X \star \mathscr{F}Y \\
\varphi \downarrow & \text{(A)} & \downarrow \gamma \\
\mathscr{F}(X \diamond Y) & \xrightarrow[\mathscr{F}(\sigma)]{} & \mathscr{F}(X \star Y)
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{F}X \diamond \mathscr{F}Y & \xrightarrow{\tau} & \mathscr{F}Y \star \mathscr{F}X \\
\varphi \downarrow & \text{(B)} & \downarrow \gamma \\
\mathscr{F}(X \diamond Y) & \xrightarrow[\mathscr{F}(\tau)]{} & \mathscr{F}(Y \star X)
\end{array}
$$

When $\mathscr{C}$ is the unit duoidal category, this recovers the result of Garner and López Franco.

Eckmann-Hilton converse

Theorem (A.-Cóppola).
Let $\mathscr{C}$ and $\mathscr{D}$ be normal duoidal categories.
Let $(\mathscr{F}, \varphi, \gamma) : \mathscr{C} \to \mathscr{D}$ be such that:

- both $(\mathscr{F}, \varphi)$ and $(\mathscr{F}, \gamma)$ are monoidal,
- the transformation $\mathscr{F}(\sigma \star \sigma)$ is monic,
- axioms (A) and (B) hold.

Then $(\mathscr{F}, \varphi, \gamma)$ is duoidal.

Back to the source:
an operation $\bullet$ that is both commutative and associative satisfies

$$(a \bullet b) \bullet (c \bullet d) = (a \bullet c) \bullet (b \bullet d).$$

APPLICATION: HOCHSCHILD COHOMOLOGY

Recall: the Hochschild complex

Let $\mathbb{k}$ be a commutative ring, A a $\mathbb{k}$-algebra, and M an A-bimodule.
The Hochschild cochain complex $\mathscr{H}(A, M)$ is (in degree n)

$$\mathscr{H}^n(A, M) = \mathrm{Hom}_{\mathbb{k}}(A^{\otimes n}, M).$$

The differential involves the product of A and the A-actions on M.

Let $A \to B$ be a morphism of $\mathbb{k}$-algebras. View B as an A-bimodule by restriction.
Then the cup product of cochains is defined:
given $f \in \mathscr{H}^p(A, B)$ and $g \in \mathscr{H}^q(A, B)$, their cup product is

$$f \smile g : A^{\otimes(p+q)} = A^{\otimes p} \otimes A^{\otimes q} \xrightarrow{f \otimes g} B \otimes B \to B.$$

- We want to analyze the potential commutativity of this product at the cochain level.
- Under the present assumptions, the product need not be commutative, not even at the cohomological level, and not even if the algebra B is commutative.

Recall: commutativity of the cup product

Let H be a $\mathbb{k}$-bialgebra.
View $\mathbb{k}$ as a trivial H-bimodule.

Fact. The cup product on the Hochschild complex $\mathscr{H}(\mathrm{H}, \mathbb{k})$ is commutative up to homotopy:

$$f \smile g \equiv (-1)^{|f|\,|g|} g \smile f.$$

We will explain this fact from the perspective of the Eckmann-Hilton argument.

We will generalize it:

- By extending it to the setting of duoidal categories.
- By (suitably) extending it from the cohomological level to the cosimplicial and cochain levels.
- By extending it to a statement about the Hochschild cochain functor.

Generalization

Let $(\mathscr{C}, \diamond, \star)$ be a $\mathbb{k}$-linear duoidal category.

Let A be a monoid and M an A-bimodule in $(\mathscr{C}, \diamond)$.
The Hochschild cochain complex $\mathscr{H}(A, M)$ is defined in the usual manner.

Suppose B is a monoid in $(\mathscr{C}, \diamond)$.
Then the cup product on $\mathscr{H}(A, B)$ is defined in the usual manner.

We generalize the commutativity of the cup product to Hochschild cohomology of a bimonoid with coefficients in a duoid.

Proposition (A.-Cóppola)
Let H be a bimonoid and D a duoid in $(\mathscr{C}, \diamond, \star)$.
View D as a trivial H-bimodule via

$$H \xrightarrow{\delta_0} K \xrightarrow{\nu_0} D.$$

The cup product on $\mathscr{H}(H, D)$ is commutative up to homotopy.

We will derive this result from properties of the Hochschild functor.

Preliminary: bimodules

Let $(\mathscr{C}, \otimes)$ be a monoidal category.

Let A be a monoid in $(\mathscr{C}, \otimes)$.
Let $\mathscr{C}_A$ denote the category of A-bimodules M in $\mathscr{C}$:

$$A \otimes M \xrightarrow{\lambda} M, \quad M \otimes A \xrightarrow{\rho} M.$$

Suppose all reflexive coequalizers exist in $\mathscr{C}$ and are preserved by $X \otimes (-)$ and $(-) \otimes X$ for each X in $\mathscr{C}$. Then:

- $(\mathscr{C}_A, \otimes_A)$ is monoidal with $\otimes_A$ defined by

$$M \otimes H \otimes N \underset{\mathrm{id} \otimes \lambda}{\overset{\rho \otimes \mathrm{id}}{\rightrightarrows}} M \otimes N \longrightarrow M \otimes_A N.$$

Suppose now $(\mathscr{C}, \diamond, \star)$ is duoidal and H is a bimonoid therein. Then:

- If M and N are H-bimodules in $(\mathscr{C}, \diamond)$, so is $M \star N$ under

$$H \diamond (M \star N) \xrightarrow{\delta \diamond \mathrm{id}} (H \star H) \diamond (M \star N) \xrightarrow{\zeta} (H \diamond M) \star (H \diamond N) \to M \star N.$$

- $(\mathscr{C}_H, \diamond_H, \star)$ is duoidal.

Preliminary: cochain complexes

Let $\mathscr{E}$ be the category of cochain $\mathbb{k}$-complexes:

$$C = (C^n, d^n).$$

For each $n \geq 0$, C^n is a $\mathbb{k}$-module, and

$$d^n : C_n \to C_{n+1}$$

is a collection of $\mathbb{k}$-module morphisms such that $d^{n+1}d^n = 0$.

The category $\mathscr{E}$ is symmetric monoidal under

$$(C \bullet D)^n = \bigoplus_{i=0}^{n} C^i \otimes D^{n-i},$$

$$d^n(x \otimes y) = d^i(x) \otimes y + (-1)^i x \otimes d^{n-i}(y)$$

$$\text{for } x \in C^i,\ y \in D^{n-i}.$$

Recall the operations $\diamond$ and $\star$ for cosimplicial modules.

Note that $\diamond$ resembles $\bullet$, but $\star$ (defined diagonally) does not.

The Hochschild complex of a bimonoid

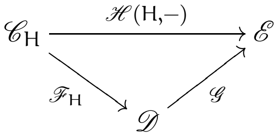
Let H be a bimonoid in $(\mathscr{C}, \diamond, \star)$.

Consider the following duoidal categories:

- $\mathscr{C}_{\mathrm{H}}$, the category of H-bimodules in $(\mathscr{C}, \diamond)$.
- $\mathscr{D}$, the category of cosimplicial $\mathbb{k}$-modules.
- $\mathscr{E}$, the category of cochain complexes of $\mathbb{k}$-modules.

They are all duoidal. $\mathscr{D}$ is normal, while $\mathscr{E}$ is symmetric monoidal.

The Hochschild functor $\mathscr{H}(\mathrm{H}, -)$ factors:



The functor $\mathscr{F}_H : \mathscr{C}_H \to \mathscr{D}$

The degree n component of $\mathscr{F}_H(M)$ is

$$\mathrm{Hom}_{\mathscr{C}}(H^{\diamond n}, M).$$

Proposition (A.-Cóppola).
$\mathscr{F}_H$ carries two monoidal structures:

$$\mathscr{F}_H M \diamond \mathscr{F}_H N \xrightarrow{\varphi} \mathscr{F}_H(M \diamond_H N), \quad \mathscr{F}_H M \times \mathscr{F}_H N \xrightarrow{\gamma} \mathscr{F}_H(M \star N).$$

- φ is concatenation of cochains.
- γ is convolution of cochains using $\delta : H \to H \star H$.

With these structures, $(\mathscr{F}_H, \varphi, \gamma)$ is duoidal.

Note how we employed the bimonoid structure of H.

The functor $\mathscr{G} : \mathscr{D} \to \mathscr{E}$

This functor simply builds the differential from the coface maps:

$$(X^n, d^i, s^j) \overset{\mathscr{G}}{\longmapsto} (X^n, \sum_{i=0}^{n+1} (-1)^i d^i).$$

Proposition (A.-Cóppola).

- $\mathscr{G}$ carries 3 monoidal structures:

$$\mathscr{G}X \bullet \mathscr{G}Y \xrightarrow{\varphi} \mathscr{G}(X \diamond Y), \quad \mathscr{G}X \bullet \mathscr{G}Y \xrightarrow{\mathrm{AW}} \mathscr{G}(X \times Y), \quad \mathscr{G}X \bullet \mathscr{G}Y \xrightarrow{\widetilde{\mathrm{AW}}} \mathscr{G}(X \times Y).$$

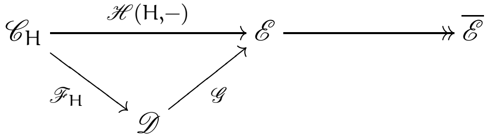
  - φ is strong.
  - AW and $\widetilde{\mathrm{AW}}$ are the two versions of Alexander-Whitney (front-back and back-front).

- $(\mathscr{G}, \varphi, \mathrm{AW})$ satisfies axiom (A), while $(\mathscr{G}, \varphi, \widetilde{\mathrm{AW}})$ satisfies axiom (B).

This last statement is the crux of the matter.

Up to homotopy

Classical facts. Dold: AW and $\widetilde{\mathrm{AW}}$ coincide up to homotopy.
Eilenberg-Zilber: AW is invertible up to homotopy.



Corollary. (A.-Cóppola).

- The functor $\mathscr{D} \xrightarrow{\mathscr{G}} \mathscr{E} \twoheadrightarrow \overline{\mathscr{E}}$ is duoidal.
- The functor $\mathscr{C}_{\mathrm{H}} \xrightarrow{\mathscr{H}(\mathrm{H},-)} \mathscr{E} \twoheadrightarrow \overline{\mathscr{E}}$ is duoidal.

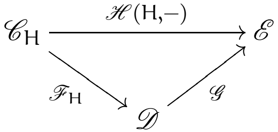
Proof. When we pass to $\overline{\mathscr{E}}$, axioms (A) and (B) hold for AW by Dold.
Then we apply the EH converse for duoidal functors.
(We use EZ for the hypothesis on monomorphisms.)

Commutativity of the cup product

Lemma (A.-Cóppola).Let H be a bimonoid and D a duoid in $(\mathscr{C}, \diamond, \star)$. View D as a trivial H-bimodule via

$$\mathrm{H} \xrightarrow{\delta_0} \mathrm{K} \xrightarrow{\nu_0} \mathrm{D}.$$

Then D is a duoid in $\mathscr{C}_{\mathrm{H}}$. Moreover, every duoid in $\mathscr{C}_{\mathrm{H}}$ is of this form.

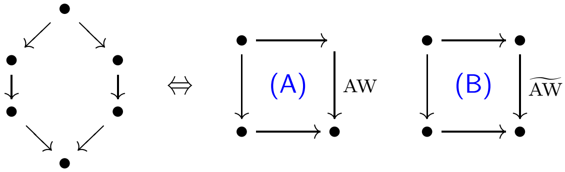


Proposition (A.-Cóppola). Let H and D be as above. Then the Hochschild cochain complex $\mathscr{H}(\mathrm{H}, \mathrm{D})$ carries two products which, up to homotopy, coincide and are commutative.

Proof. The duoidal functor $\mathscr{C}_{\mathrm{H}} \xrightarrow{\mathscr{H}(\mathrm{H},-)} \mathscr{E} \to \overline{\mathscr{E}}$ sends the duoid D in $\mathscr{C}_{\mathrm{H}}$ to a duoid $\mathscr{H}(\mathrm{H}, \mathrm{D})$ in $\overline{\mathscr{E}}$. Since the latter category is braided monoidal, Eckmann-Hilton applies.

Comments

- The classical proofs of the commutativity of the cup product (at the level of cohomology) rely on Dold's fact about AW. Our proof of commutativity (up to homotopy of cochains) does too.
- We obtain the more general property that the Hochschild functor $\mathscr{H}(\mathrm{H}, -)$ is duoidal, up to homotopy.
- The initial obstacle for proving this is: how to prove that a noncommutative diagram actually commutes up to homotopy? Constructing explicit chain-homotopies is challenging.
- We circumvent this by applying the EH converse for duoidal functors. Diagrams (A) and (B) do commute on the nose. They break the lack of commutativity into two different pieces that agree up to homotopy, by Dold.



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