

Demystifying Codensity Monads via Duality

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Topos Colloquium, May 2026

Recap: Monads Model Computational Effects

Kleisli cat

$$X \longrightarrow TY$$

"maps with Π -effect"

$$\frac{X \xrightarrow{f} TY \quad Y \xrightarrow{g} TZ}{X \xrightarrow{g \circ f} TZ}$$

Syntax

Monad Π

$$\Pi: \text{Set} \longrightarrow \text{Set}$$

$$\eta_x: X \longrightarrow \Pi X$$

$$\frac{f: X \longrightarrow TY}{f^*: \Pi X \longrightarrow TY}$$

$$f^*: \Pi X \longrightarrow TY$$

+ 3 natural axioms:

$$- \eta_x^* = \text{id}_{\Pi X}$$

$$- f^* \cdot \eta_x = f$$

$$- f^* \cdot g^* = (f^* \cdot g)^*$$

Eilenberg-Moore cat

Algebras

$$\Pi A \longrightarrow A$$

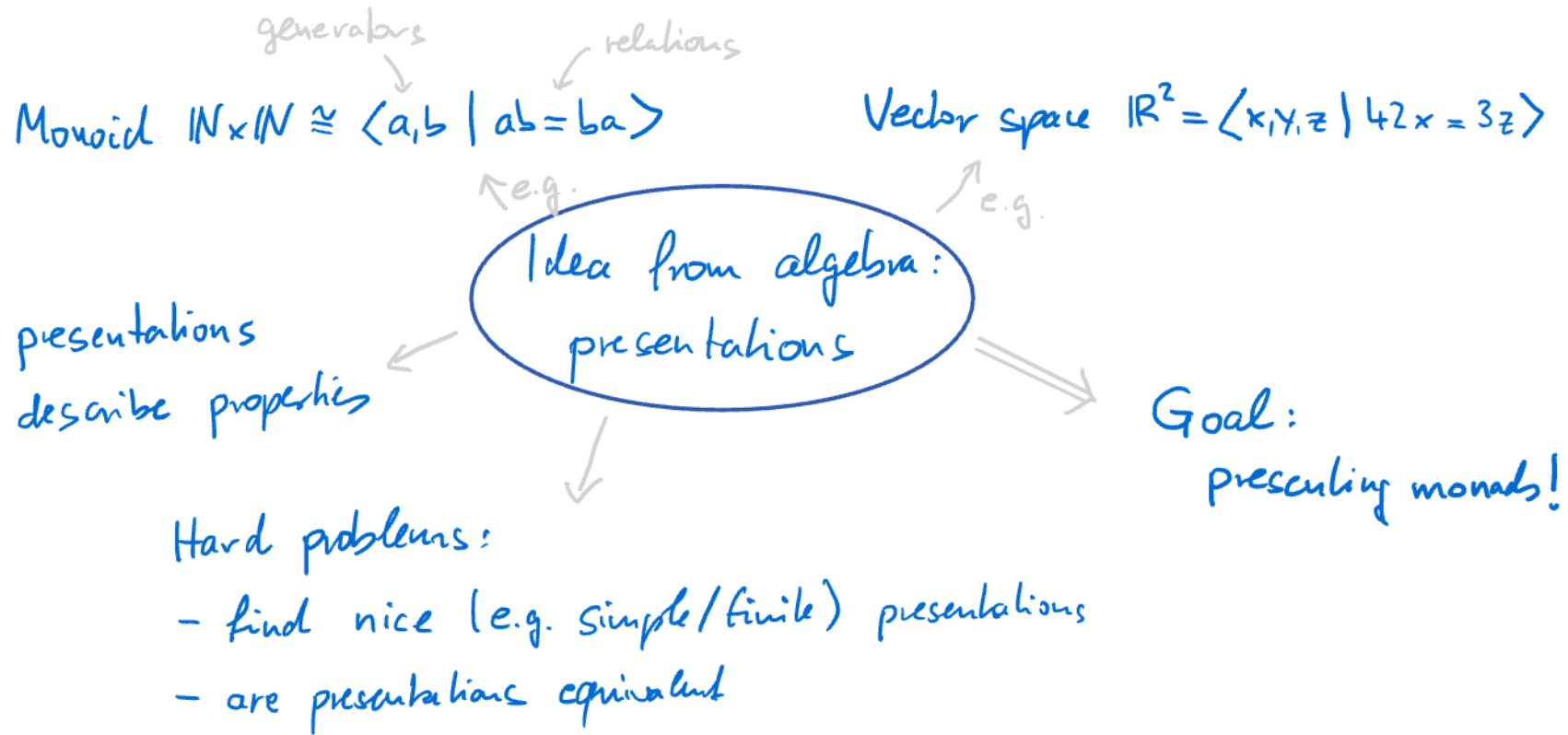
"evaluate Π -effects"

Semantics

Examples of Monads

TX	Computations	Algebras
$\mathcal{P}_f X = \{S \subseteq X \mid S \text{ finite}\}$	nondeterminism	semilattices
$X^* = \{x_1 x_2 \dots x_n \mid x_i \in X\}$		monoids
$\mathcal{D}X = \{\sum_{i=1}^n r_i x_i \mid r_i \in [0,1], x_i \in X, \sum_{i=1}^n r_i = 1\}$	probabilities	convex sets
$2^{2^X} = \{f: (X \rightarrow 2) \rightarrow 2\}$		complete atomic Boolean algs
$\mathcal{U}X = \{U \subseteq 2^X \mid U \text{ ultrafilter}\}$		compact Hausdorff spaces
$\mathbb{E}X = \{\text{fin. add. prob. meas. } 2^X \rightarrow [0,1]\}$	"probabilistic continuations"	observable convex compact Hausdorff sp.

Leinster: Where do Monads come from?



Examples of Monads

TX

Computations

Algebras

$$\mathcal{P}_f X = \{S \subseteq X \mid S \text{ finite}\}$$

nondeterminism

semilattices

$$X^* = \{x_1 x_2 \dots x_n \mid x_i \in X\}$$

monoids

$$\mathcal{D}X = \left\{ \sum_{i=1}^n r_i x_i \mid r_i \in [0,1], x_i \in X, \sum_{i=1}^n r_i = 1 \right\}$$

probabilities

convex sets

finitary presentations by operations & equations

↳ e.g. $(-)^*$... operations $e/0, \cdot/2$

equations $x \cdot e = e \cdot x = x$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Examples of Monads

TX

Computations

Algebras

involved in finitary presentations

↖ class of operations & equations
unbounded arity

$$2^{2^X} = \{ f: (X \rightarrow 2) \rightarrow 2 \}$$

$$\mathcal{U}X = \{ U \subseteq 2^X \mid U \text{ ultrafilter} \}$$

$$\mathbb{E}X = \{ \text{fin. add. prob. meas. } 2^X \rightarrow [0,1] \}$$

"probabilistic
continuations"

complete atomic Boolean algebras

compact Hausdorff spaces

observable convex
compact Hausdorff sp.

Recap: Right Kan Extension

Idea: "best approximation" to extending the domain of F through K :

$$\begin{array}{ccc} & K \rightarrow & \mathcal{D} \\ \mathcal{C} & \searrow & \vdots \\ & F \rightarrow & \mathcal{E} \end{array} \quad \begin{array}{l} \Downarrow \varepsilon \\ \text{Ran}_K F \\ \downarrow \end{array} \quad \mathcal{E} \text{ complete}$$

Definition:

$$[\mathcal{C}, \mathcal{E}] \begin{array}{c} \xleftarrow{- \circ K} \\ \xrightarrow{\perp} \\ \xrightarrow{\text{Ran}_K(-)} \end{array} [\mathcal{D}, \mathcal{E}]$$

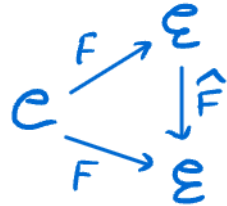
Exists for \mathcal{C} small and \mathcal{E} complete:

$$(\text{Ran}_K F) \mathcal{D} = \int_{\mathcal{C} \in \mathcal{E}} \mathcal{D}(\mathcal{D}, K\mathcal{C}) \wedge FC = \lim_{\substack{\mathcal{D} \rightarrow K\mathcal{C} \\ \mathcal{C} \in \mathcal{E}}} FC$$

Codensity Presentations

Kock 1966: Every functor $\mathcal{C} \xrightarrow{\text{small}} \mathcal{E} \xrightarrow{\text{complete}}$ induces the codensity monad \hat{F} of F :

$$\hat{F} = \text{Ran}_F F, \quad \hat{F}X = \lim_{X \rightarrow FE} FE$$



✓ F is the codensity presentation of \hat{F}

✓ properties of \hat{F} derive from properties of F Kock '66

Canonical Codensity Presentation

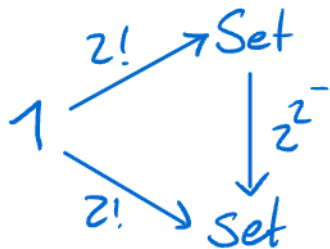
- codensity monads generalize monads from adjoints:
if $L \dashv R$, then $\hat{R} \cong RL$

- every monad T has a codensity presentation:

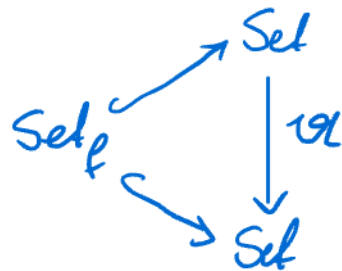
e.g. $T \cong \hat{U}_T$, $\text{Kl}(T) \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{U_T} \end{array} \mathcal{C}$

Goal: Find a simple presentation for a given monad! \blacktriangleright

Examples

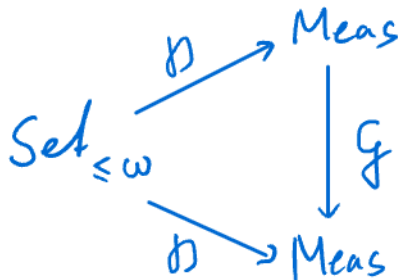


continuation monad



ulhafilke monad

Kennison + Gildenhuis '71, Leinster '16

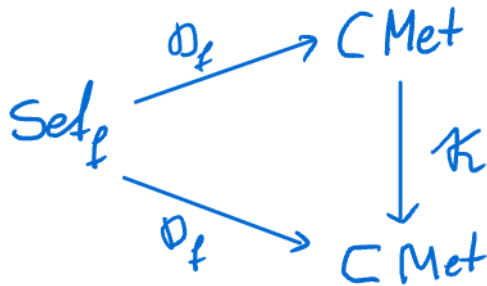


Giry monad

Avery '16, von Belle '22

⊗ Proofs are involved

also Shirazi '24



Kantorovich monad

van Belle '22

Our Contribution

Reduce Proofs $T = \hat{F}$ to two ingredients:

Codensity Monad = Density + Duality

✓ Capture + simplify all instances in the literature

↳ Kock '66; Avery '16; Leinster '16; Sipos '18; Gehrke et al. '20; Reggio '20; Adámek, Sousa '21; van Belle '22; Shirazi '24

✓ Several new presentations for important monads

Recap: Dense Functor

Given $G: \mathcal{A} \rightarrow \mathcal{B}$ consider $\pi_B: G \downarrow \mathcal{B} \rightarrow \mathcal{A}$

G dense if $\mathcal{B} = \text{colim}(G \cdot \pi_B)$

$$\begin{array}{ccc} GA & \xrightarrow{f} & B \\ G \downarrow h & & \downarrow h \\ GA' & \xrightarrow{f'} & B \end{array} \quad \begin{array}{c} \dashrightarrow \\ \dashrightarrow \\ \dashrightarrow \end{array} \quad \begin{array}{c} A \\ \downarrow h \\ A' \end{array}$$

E.g. $\{1\} \hookrightarrow \text{Set}$, $\text{Set}_f \hookrightarrow \text{Set}$

$\text{Alg}_T \hookrightarrow \text{Alg}$ — variety of finitary algebras

$\text{fil}(T) \hookrightarrow \text{EM}(T)$ for $T: \text{Set} \rightarrow \text{Set}$

$\text{fil}_f(T) \hookrightarrow \text{EM}(T)$ if T is finitary

Main Theorem

$$\begin{array}{ccc}
 \mathcal{C}_0 & \xrightarrow{F} & \mathcal{C} \begin{array}{c} \leftarrow \hat{F} \\ \leftarrow \end{array} \\
 \uparrow E & & \uparrow L \quad \downarrow R \\
 \mathcal{D}_0^{op} & \xrightarrow{G^{op}} & \mathcal{D}^{op}
 \end{array}$$

& G dense

Matteo TAC'25 proved different formulation



$$\Rightarrow RL \cong \hat{F}$$

"codensity setting"

Proof:

$$RLX \cong \begin{array}{c} \text{G dense} \\ \swarrow \\ R(\operatorname{colim}_{f:GD \rightarrow LX} GD) \end{array} \cong \begin{array}{c} \text{Preserves limits} \\ \swarrow \\ \lim_{f:GD \rightarrow LX} RGD \end{array}$$

$$\begin{array}{c} L \dashv R \\ \swarrow \\ \cong \lim_{g:X \rightarrow RGD} RGD \end{array} \cong \begin{array}{c} \lim_{g:X \rightarrow FED} FED \\ \swarrow \\ \cong \lim_{g:X \rightarrow FC} FC \end{array}$$

$$\begin{array}{c} \text{limit formula for } \operatorname{Ran} \\ \swarrow \\ \cong (\operatorname{Ran}_F F) X = \hat{F} X \end{array} \quad \begin{array}{c} RG^{op} \cong FE \\ \swarrow \\ \cong \end{array} \quad \begin{array}{c} E \text{ equivalence} \\ \swarrow \\ \cong \end{array}$$

□

Canonical Codensity Setting

Every codensity monad has a codensity setting:

$$\begin{array}{ccc}
 \mathcal{C}_0 & \xrightarrow{F} & \mathcal{C} \xrightarrow{\hat{F}} \\
 \parallel & & \uparrow \text{N} \left(\dashv \right) \text{R} \\
 (\mathcal{C}_0^{\text{op}})^{\text{op}} & \xrightarrow{\gamma^{\text{op}}} & [\mathcal{C}_0, \text{Set}]^{\text{op}}
 \end{array}$$

conerve - totalization adjunction

$$NC = \mathcal{C}(\mathcal{C}, F(-))$$

$$RC = \text{Ran}_{\tilde{\gamma}} F$$

$$\tilde{\gamma} : \mathcal{C}_0^{\text{op}} \longrightarrow [\mathcal{C}_0, \text{Set}]$$

$$\hat{F} \cong RN$$

Leinster '13

contra variant
Yoneda embedding

BUT: This does not help to find a simple codensity presentation of a given monad.

Two ingredients

① Dual adjunctions restricting to simple dualities

$$\begin{array}{ccc}
 \mathcal{C}_0 & \xrightarrow{F} & \mathcal{C} \overset{\hat{F}}{\curvearrowright} \\
 |s & & \downarrow L \uparrow R \\
 \mathcal{D}_0^{\text{op}} & \xrightarrow{G^{\text{op}}} & \mathcal{D}^{\text{op}}
 \end{array}$$

topology sets, topological sp., measurable sp.

vs.

algebra Boolean, frames, co-Boolean algs

well studied



② Compute density instead of codensity

$X \in \mathcal{D}$ "determined" by G -subobjects



- $\mathbb{Q} \hookrightarrow \mathbb{R}$ dense
- $\{1\} \hookrightarrow \text{Set}$ dense
- $\{a, b\}^* \hookrightarrow \text{Mon}$ dense

formally dual, but "less natural"

e.g. $\{2\} \in \text{Stone}$

$\{\mathbb{S}^1 \times \mathbb{S}^1\} \in \text{AbGrp}(\text{CHaus})$

The Power of Category Theory

Clarity: separate generic from domain-specific parts

• $\widehat{\text{Set}_f} \hookrightarrow \text{Set} \cong \mathcal{U}$ is "completely categorical"

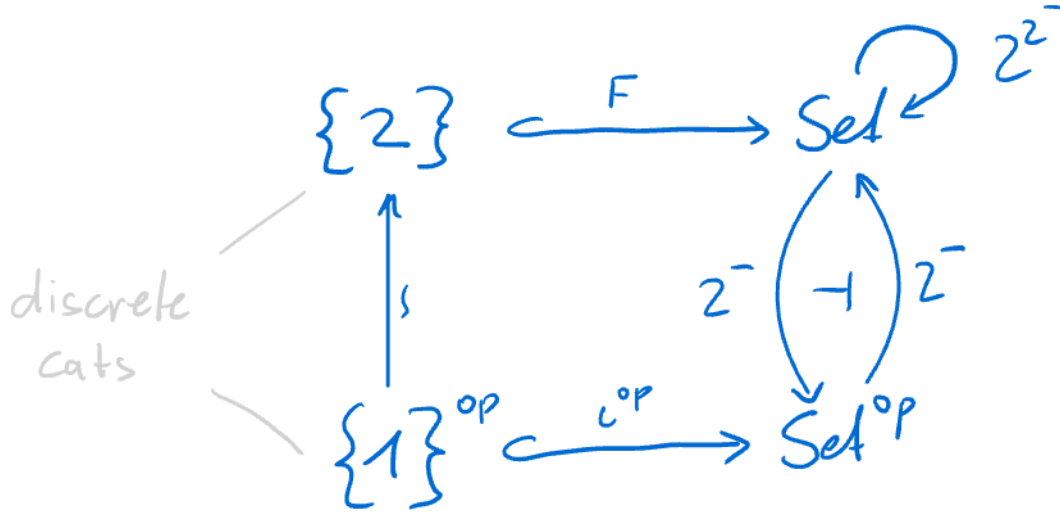
• $\widehat{\text{Mark}_\omega} \hookrightarrow \text{Meas} \cong \text{Giry monad}$ uses integral representation theorem

↑
countably (discrete)
Markov kernels

↑
measurable
spaces

↑
probability
distributions

Example: Neighbourhood Monad

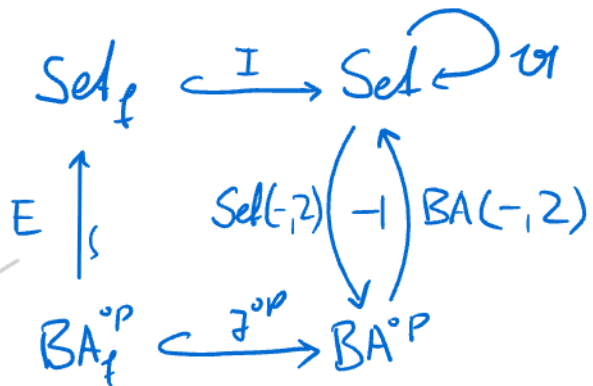


$$z^1 = 2 \quad \checkmark$$

$$\{1\} \hookrightarrow Set \text{ dense} \quad \checkmark$$

$$\Rightarrow \hat{F} = z^{z^-}$$

Example: Ultrafilter Monads



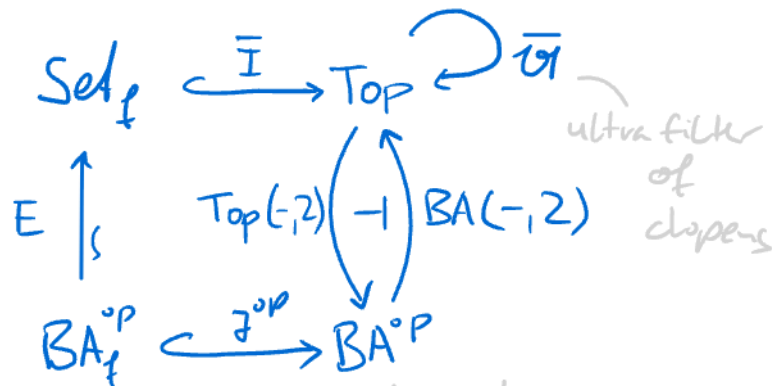
Birkhoff duality

$$I \cdot E \cong \text{BA}(J^{\circ p}(-), 2) \quad \checkmark$$

$$\text{BA}_f \xrightarrow{J} \text{BA} \text{ dense} \quad \checkmark$$

$$\Rightarrow \hat{I} \cong \mathcal{U}$$

— Kennison & Goldenhuys '71



discuss

$$\text{Top}(X, 2) = \text{clopens of } X$$

$$\Rightarrow \checkmark \quad J \text{ dense} \quad \checkmark$$

$$\Rightarrow \hat{\bar{I}} = \bar{\mathcal{U}}$$

— Sipos '18

Example: Canonical Extension (Turning the Square)

$$\begin{array}{ccc}
 \text{BA}_f & \xrightarrow{J} & \text{BA} \xrightarrow{\text{Z}^{\text{rel}(-)}} \\
 \uparrow E' & & \uparrow \text{BA}(-,2) \left(\begin{array}{c} \leftarrow \\ \rightarrow \end{array} \right) \text{Set}(-,2) \\
 \text{Set}_f^{\text{op}} & \xrightarrow{I^{\text{op}}} & \text{Set}^{\text{op}}
 \end{array}$$

$$J \cdot E' \cong \text{Set}(I^{\text{op}}(-), 2) \quad \checkmark$$

$$\text{Set}_f \xrightarrow{I} \text{Set} \text{ dense} \quad \checkmark$$

$$\Rightarrow \hat{J} \cong \underbrace{\text{Z}^{\text{rel}(-)}}_{\text{new}}$$

Filter Monads

Example: Filter Monads

$$\begin{array}{ccc}
 \text{MSL}_f & \xrightarrow{u} & \text{Set} \left\langle \begin{array}{c} \curvearrowright \\ \mathcal{F} \end{array} \right. \\
 \uparrow \scriptstyle z^- \uparrow_s & & \downarrow \scriptstyle z^- \left(\begin{array}{c} \uparrow \\ - \\ \downarrow \end{array} \right) \downarrow z^- \Rightarrow \hat{U} \cong \mathcal{F} \\
 \text{MSL}_f^{\text{op}} & \xrightarrow{\mathcal{J}^{\text{op}}} & \text{MSL}^{\text{op}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{Rel}_f = \text{Ker}_f(\mathcal{P}_f) & \xrightarrow{U_{\mathcal{P}_f}} & \text{Set} \left\langle \begin{array}{c} \curvearrowright \\ \mathcal{F} \end{array} \right. \\
 \uparrow \scriptstyle (-)^{\text{op}} \uparrow_s & & \downarrow \scriptstyle z^- \left(\begin{array}{c} \uparrow \\ - \\ \downarrow \end{array} \right) \downarrow z^- \Rightarrow \hat{U}_{\mathcal{P}_f} \cong \mathcal{F} \\
 \text{Rel}_f^{\text{op}} = \text{Ker}_f(\mathcal{P}_f)^{\text{op}} & \xrightarrow{I_{\mathcal{P}_f}^{\text{op}}} & \text{EM}(\mathcal{S}_f)^{\text{op}} = \text{MSL}^{\text{op}}
 \end{array}$$

Variation: topological filters

$\bar{\mathcal{S}}X$ = filter of opensets of X

$\text{MSL}(X, \mathcal{Z}) \subseteq \mathcal{S}^{|\mathcal{M}|}$ top. sp.

$V_{\mathcal{S}_f} X = \mathcal{S}^X$ *Sierpinski space*

$$\Rightarrow \hat{V}_{\mathcal{S}_f} \cong \mathcal{F}$$

$$\begin{array}{ccc}
 \text{Ker}_f(\mathcal{P}_f) & \xrightarrow{V_{\mathcal{S}_f}} & \text{Top} \left\langle \begin{array}{c} \curvearrowright \\ \bar{\mathcal{F}} \end{array} \right. \\
 \uparrow \scriptstyle (-)^{\text{op}} \uparrow_s & & \downarrow \scriptstyle \text{Top}(-, \mathcal{S}) \left(\begin{array}{c} \uparrow \\ - \\ \downarrow \end{array} \right) \downarrow \text{MSL}(-, \mathcal{Z}) \\
 \text{Ker}_f(\mathcal{P}_f)^{\text{op}} & \xrightarrow{I_{\mathcal{S}_f}^{\text{op}}} & \text{MSL}^{\text{op}}
 \end{array}$$

Double Dualization Monads

Example: Double Dualization Monads

$$\begin{array}{ccc}
 \mathbf{BA}_f & \xrightarrow{u} & \mathbf{Set}^{\circlearrowleft} \\
 \uparrow s & & \downarrow z^{-} \left(\begin{array}{c} \uparrow \\ - \\ \downarrow \end{array} \right) z^{-} \\
 \mathbf{Set}_f & \xrightarrow{I^{op}} & \mathbf{Set}^{op}
 \end{array}$$

$$\Rightarrow \hat{U} \cong z^{z^{-}}$$

← new

$$\begin{array}{ccc}
 \mathbf{MSL}_f & \xrightarrow{I} & \mathbf{MSL}^{\circlearrowleft} \\
 \uparrow s & & \downarrow z^{-} \left(\begin{array}{c} \uparrow \\ - \\ \downarrow \end{array} \right) z^{-} \\
 \mathbf{MSL}_f^{op} & \xrightarrow{I^{op}} & \mathbf{MSL}^{op}
 \end{array}$$

$$\Rightarrow \hat{I} = \mathbf{MSL}(\mathbf{MSL}(-, z), z)$$

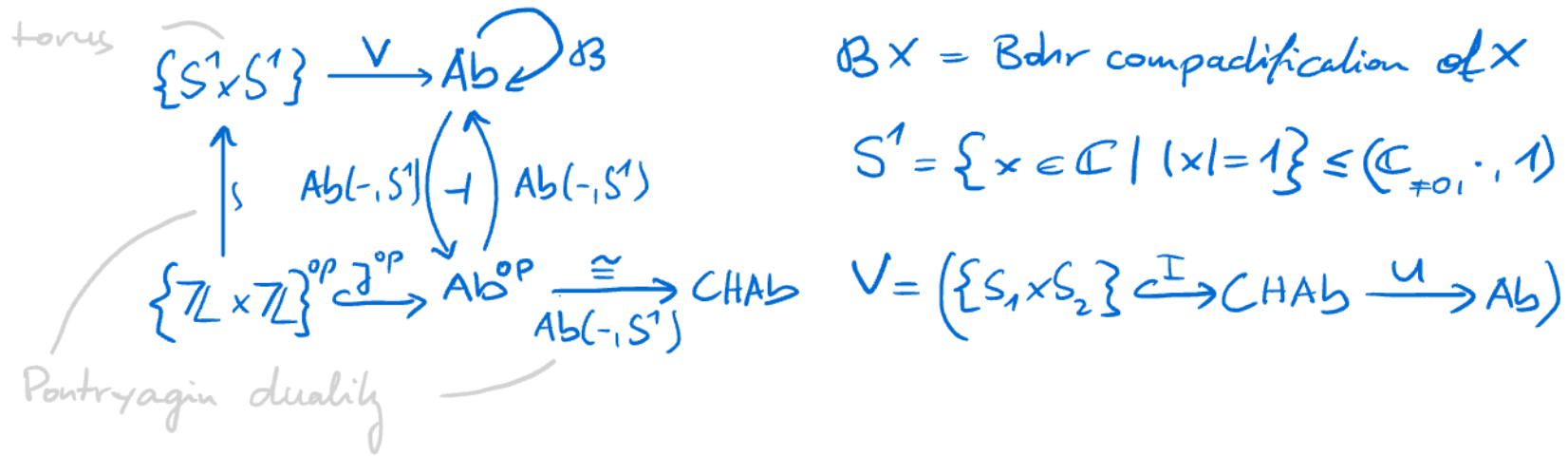
Adámek, Sousa Adv. Math '21

$$\begin{array}{ccc}
 \mathbf{Vec}_{fd} & \xrightarrow{I} & \mathbf{Vec}^{\circlearrowleft} \text{ } (-)^{**} \\
 \uparrow s & & \downarrow (-)^* \left(\begin{array}{c} \uparrow \\ - \\ \downarrow \end{array} \right) (-)^* \\
 \mathbf{Vec}_{fd}^{op} & \xrightarrow{I^{op}} & \mathbf{Vec}
 \end{array}$$

$$\Rightarrow \hat{I} \cong (-)^{**}$$

Leinster TAC '13

Example: A more involved double dualization monad



$$\mathcal{D} : S^1 \times S^1 \cong \text{Ab}(\mathbb{T}, S_1) \times \text{Ab}(\mathbb{T}, S_1) \cong \text{Ab}(\mathbb{T} \times \mathbb{T}, S_1) \quad \checkmark$$

$$\mathcal{J} : \{ \mathbb{T} \times \mathbb{T} \} \xrightarrow{\mathcal{J}} \text{Ab} \text{ dense} \quad \checkmark$$

full subcat

$$\Rightarrow \hat{\mathcal{V}} \cong \mathcal{B}$$

new

The Monad of Isbell Duality

Example: Isbell Duality

\sim is the dual adjunction between contra- & covariant presheaves

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\gamma} & [\mathcal{C}^{op}, \text{Set}] \\
 \parallel & & \circlearrowleft \left(+ \right)_{\text{Spec}} \\
 (\mathcal{C}^{op})^{op} & \xrightarrow{\tilde{\gamma}^{op}} & [\mathcal{C}, \text{Set}]^{op}
 \end{array}$$

$$\gamma: \mathcal{C} \mapsto \mathcal{C}(-, C) \quad \text{Yoneda}$$

$$\tilde{\gamma}: \mathcal{C} \mapsto \mathcal{C}(C, -) \quad \text{embeddings}$$

$$\mathcal{O}X: (\mathcal{C} \mapsto \text{Nat}(X, \mathcal{C}(-, C)))$$

$$\text{Spec } X: (\mathcal{C} \mapsto \text{Nat}(X, \mathcal{C}(C, -)))$$

$$\begin{aligned}
 \Downarrow \text{ by Yoneda Lemma: } \text{Spec}(\tilde{\gamma}(D))(C) &= \text{Nat}(\mathcal{C}(D, -), \mathcal{C}(C, -)) \\
 &\cong \mathcal{C}(D, C) = \gamma(D)(C) \quad \checkmark
 \end{aligned}$$

$\tilde{\gamma}$ is dense \checkmark

$$\Rightarrow \hat{\tilde{\gamma}} \cong \text{Spec} \cdot \mathcal{O} \quad \text{Koch '66, Di Liberti JPAA'20}$$

Monads on top. Spaces

Example: Vietoris Monad

$\mathbb{V}X = \{C \mid C \subseteq X, C \text{ closed}\}$ with "hit-&-miss" topology given by all

$U \subseteq X$ open $\left\{ \begin{array}{l} \diamond U = \{C \in \mathbb{V}X \mid C \cap U \neq \emptyset\} \\ \square U = \{C \in \mathbb{V}X \mid C \subseteq U\} \end{array} \right.$ "C hits U"
"C misses X \setminus U"

$$\text{Rel}_f \cong \text{Kl}_f(\mathcal{P}_f) \xrightarrow{U_{\mathcal{P}_f}} \text{Stone} \leftarrow \mathbb{V}$$

$$\begin{array}{c} \uparrow \\ (-)^{\text{op}} \end{array}$$

$$\text{Stone}(-, 2) \left(\begin{array}{c} \uparrow \\ - \\ \downarrow \end{array} \right) \text{JSL}(-, 2)$$

$$\Rightarrow \hat{U}_{\mathcal{P}_f} \cong \mathbb{V}$$

$$\text{Rel}_f \cong \text{Kl}_f(\mathcal{P}_f) \xrightarrow{I^{\text{op}}} \text{EM}(\mathcal{P}_f) = \text{JSL}^{\text{op}}$$

Gehrke, Petrişan, Reggio
MSCS'20

Variants:



lower Vietoris monad - new

sobrification monad - Sipos Math. Slovak. '18

Probability Monads

Example: The Expectation Monad

$$\mathcal{D}X = \left\{ \sum_{i=1}^{\infty} r_i x_i \mid r_i \in [0,1], x_i \in X, \sum_{i=1}^{\infty} r_i = 1 \right\} \quad \mathbb{E}X = \{ \text{fin. add. prob. meas. } \mathbb{Z}^X \rightarrow [0,1] \}$$

finite stochastic matrices

$$\text{Mark}_f \cong \text{KL}_f(\mathcal{D}) \xrightarrow{U_0} \text{Set} \xrightarrow{\mathbb{E}}$$

restricts

$$\text{BA}_f \xrightarrow{\sim} \text{Set}_f$$

$$\begin{array}{ccc} \text{EMod}_{\text{fd}}^{\text{op}} & \xrightarrow{I^{\text{op}}} & \text{EMod}^{\text{op}} \\ \uparrow \text{res} & & \uparrow \text{EMod}(-, [0,1]) \\ \text{Set}_f & & \text{Set}(-, [0,1]) \end{array} \quad \text{EMod}(-, [0,1]) \cong \text{State}$$

Jacobs TCS'10

effect modules \cong "probabilistic vector spaces"

- \mathbb{E} \checkmark

I dense \checkmark

Staton & Uijlen 18C'18

- \mathbb{E} = monad of \dashv follows from a discrete integral repr. thm.

$$\Rightarrow \widehat{U_0} \cong \mathbb{E}$$

Jacobs et al. 18C'16

Summary

* Codensity monads \cong Duality + Density

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{F} & \mathcal{C} \begin{array}{c} \leftarrow \hat{F} \\ \uparrow \\ L \begin{array}{c} (-) \\ \downarrow \\ R \end{array} \end{array} \\ \downarrow |s & & \\ \mathcal{D}_0^{\text{op}} & \xrightarrow{G^{\text{op}}} & \mathcal{D}^{\text{op}} \end{array}$$

* all known instances arise in this way

* several new ones are obtained

Future Work

- * use our framework to study codensity presentations of other monads
e.g. variant of the Victor's monad related to compact spaces and probabilistic power domains

--- Jacobs & Furber LMCS'15

- * develop our main theorem in Ross Street's formal theory of monads
(i.e. monads in 2-categories)

- * generalize to Street's monad extensions

└ extend a monad along a 1-cell