Demystifying Codensity Monads via Duality

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Recap: Monads Model Computational Effects

Kleisli cat	Monad $\mathbb{T}$	Eilenberg-Moore cat
$X \longrightarrow TY$	$\mathbb{T}: \mathbf{Set} \longrightarrow \mathbf{Set}$	Algebras
"maps with $\mathbb{T}$-effect"	$\eta_X : X \longrightarrow \mathbb{T}X$	$\mathbb{T}A \longrightarrow A$
$\dfrac{X \xrightarrow{f} TY \quad Y \xrightarrow{g} TZ}{X \xrightarrow{g \circ f} TZ}$	$\dfrac{f: X \longrightarrow TY}{f^*: TX \longrightarrow TY}$	"evaluate $\mathbb{T}$-effects"
	+ 3 natural axioms:	
	– $\eta_X^* = id_{TX}$	
	– $f^* \cdot \eta_X = f$	
	– $f^* \cdot g^* = (f^* \cdot g)^*$	
Syntax		Semantics

Examples of Monads

TX	Computations	Algebras
$\mathcal{P}_f X = \{ S \subseteq X \mid S \text{ finite} \}$	nondeterminism	semilattices
$X^* = \{ x_1 x_2 \dots x_n \mid x_i \in X \}$		monoids
$\mathcal{D}X = \{ \sum_{i=1}^{n} r_i x_i \mid r_i \in [0,1],\ x_i \in X,\ \sum_{i=1}^{n} r_i = 1 \}$	probabilities	convex sets
$2^{2^X} = \{ f : (X \to 2) \to 2 \}$		complete atomic Boolean algs
$\mathcal{U}X = \{ U \subseteq 2^X \mid U \text{ ultrafilter} \}$		compact Hausdorff spaces
$\mathbb{E}X = \{ \text{fin. add. prob. meas. } 2^X \to [0,1] \}$	"probabilistic continuations"	observable convex compact Hausdorff sp.

Leinster: Where do Monads come from?

generators relations

Monoid $\mathbb{N}\times\mathbb{N} \cong \langle a,b \mid ab = ba \rangle$

Vector space $\mathbb{R}^2 = \langle x,y,z \mid 42x = 3z \rangle$

e.g. e.g.

Idea from algebra: presentations

presentations describe properties

Goal: presenting monads!

Hard problems:

- find nice (e.g. simple/finite) presentations
- are presentations equivalent

Examples of Monads

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$\underbrace{\qquad\qquad\qquad}$

finitary presentations by operations & equations

↳ e.g. $(-)^*$ .... operations $e/0$, $\cdot/2$

equations $x \cdot e = e \cdot x = x$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Examples of Monads

TX	Computations	Algebras

involved infinitary presentations

class of operations & equations unbounded arity

$2^{2^X} = \{ f : (X \to 2) \to 2 \}$ — Complete atomic Boolean algs

$\mathcal{U}X = \{ \mathcal{U} \subseteq 2^X \mid \mathcal{U} \text{ ultrafilter} \}$ — compact Hausdorff spaces

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Recap: Right Kan Extension

<u>Idea:</u> "best approximation" to extending the domain of F through K:

$$\begin{tikzcd} & \mathcal{D} \arrow[d, dashed, "\mathrm{Ran}_K F"] \\ \mathcal{C} \arrow[ur, "K"] \arrow[r, "F"'] \arrow[r, phantom, bend left=20, "\Downarrow\varepsilon"] & \mathcal{E} \end{tikzcd}$$

ε couniversal

<u>Definition:</u>

$$\begin{tikzcd} {[\mathcal{C},\mathcal{E}]} \arrow[r, bend right, "\mathrm{Ran}_K(-)"'] \arrow[r, phantom, "\perp"] & {[\mathcal{D},\mathcal{E}]} \arrow[l, bend right, "-\circ K"'] \end{tikzcd}$$

Exists for $\mathcal{C}$ small and $\mathcal{E}$ complete:

$$(\mathrm{Ran}_K F)D = \int_{c\in\mathcal{C}} \mathcal{D}(D,Kc) \pitchfork Fc = \lim_{\substack{D\to Kc \\ c\in\mathcal{C}}} Fc$$

Codensity Presentations

Kock 1966: Every functor $\mathcal{C} \xrightarrow{F} \mathcal{E}$ induces the *codensity monad* $\hat{F}$ of F:

small — complete

$$\boxed{\hat{F} = \mathrm{Ran}_F F \,, \quad \hat{F}X = \lim_{X \to FE} FE}$$

$$\begin{tikzcd} & \mathcal{E} \arrow[dd, "\hat{F}"] \\ \mathcal{C} \arrow[ur, "F"] \arrow[dr, "F"'] & \\ & \mathcal{E} \end{tikzcd}$$

- ✓ F is the <u>codensity presentation</u> of $\hat{F}$
- ✓ properties of $\hat{F}$ derive from properties of F — Kock '66

Canonical Codensity Presentation

- codensity monads generalize monads from adjoints:

  if $L \dashv R$, then $\hat{R} \cong RL$

- every monad T has a codensity presentation:

  e.g. $T \cong \hat{U_T}$, $\quad$ \begin{tikzcd} Kl(T) \arrow[r, bend right, "U_T"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right] \end{tikzcd}

<u>Goal</u>: Find a simple presentation for a given monad!

Examples

$$\begin{tikzcd} & \mathrm{Set} \arrow[d, "2^{2^{-}}"] \\ 1 \arrow[ur, "2!"] \arrow[r, "2!"'] & \mathrm{Set} \end{tikzcd}$$

continuation monad

$$\begin{tikzcd} & \mathrm{Set} \arrow[d, "\mathfrak{U}"] \\ \mathrm{Set}_f \arrow[ur, hook] \arrow[r, hook] & \mathrm{Set} \end{tikzcd}$$

ultrafilter monad

Kennison + Gildenhuys '71 , Leinster '16

Why?

$$\begin{tikzcd} & \mathrm{Meas} \arrow[d, "\mathcal{G}"] \\ \mathrm{Set}_{\leq \omega} \arrow[ur, "\mathcal{D}"] \arrow[r, "\mathcal{D}"'] & \mathrm{Meas} \end{tikzcd}$$

Giry monad

Avery '16 , von Belle '22

⊗ Proofs are involved

also Shirazi '24

$$\begin{tikzcd} & \mathrm{CMet} \arrow[d, "\mathcal{K}"] \\ \mathrm{Set}_f \arrow[ur, "\mathcal{D}_f"] \arrow[r, "\mathcal{D}_f"'] & \mathrm{CMet} \end{tikzcd}$$

Kantorovich monad

van Belle '22

Our Contribution

Reduce Proofs $T = \hat{F}$ to two ingredients:

$$\boxed{\text{Codensity Monad} = \text{Density} + \text{Duality}}$$

- ✓ Capture + simplify <u>all</u> instances in the literature
  - ↳ Kock '66; Avery '16; Leinster '16; Sipos '18; Gehrke et al. '20; Reggio '20; Adámek, Sousa '21; van Belle '22; Shirazi '24
- ✓ Several <u>new</u> presentations for important monads

Recap: Dense Functor

Given $G : \mathcal{A} \longrightarrow \mathcal{B}$ consider $\pi_B : G \downarrow B \longrightarrow \mathcal{A}$

$$\begin{tikzcd} GA \arrow[d, "Gh"'] \arrow[dr, "f"] & \\ GA' \arrow[r, "f'"'] & B \end{tikzcd} \longmapsto \begin{tikzcd} A \arrow[d, "h"] \\ A' \end{tikzcd}$$

G <u>dense</u> if $B = \mathrm{colim}(G \cdot \pi_B)$

E.g. – $\{1\} \longrightarrow \mathbf{Set}$, $\mathbf{Set}_f \hookrightarrow \mathbf{Set}$

– $\mathbf{Alg}_{fp} \hookrightarrow \mathbf{Alg}$ — variety of finitary algebras

– $\mathcal{K}\ell(T) \hookrightarrow \mathrm{EM}(T)$ for $T : \mathbf{Set} \to \mathbf{Set}$

– $\mathcal{K}\ell_f(T) \hookrightarrow \mathrm{EM}(T)$ if T is finitary

Main Theorem

Mateo TAC'25 proved different formulation

$$
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F"] & \mathcal{C} \arrow[loop right, "\hat{F}"] \arrow[d, bend right, "L"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{D}_0^{op} \arrow[u, "E", "\wr"'] \arrow[r, "G^{op}"] & \mathcal{D}^{op} \arrow[u, bend right, "R"']
\end{tikzcd}
\quad \& \quad G \text{ dense} \quad \Longrightarrow \quad RL \cong \hat{F}
$$

"codensity setting"

<u>Proof:</u>

$$RLX \overset{G \text{ dense}}{\cong} R\Big(\operatorname*{colim}_{f : GD \to LX} GD\Big) \overset{R \text{ preserves limits}}{\cong} \lim_{f : GD \to LX} RGD$$

$$\overset{L \dashv R}{\cong} \lim_{g : X \to RGD} RGD \overset{RG^{op} \cong FE}{\cong} \lim_{g : X \to FED} FED \overset{E \text{ equivalence}}{\cong} \lim_{g : X \to FC} FC$$

$$\overset{\text{limit formula for Ran}}{\cong} (\mathrm{Ran}_F F) X = \hat{F} X \qquad \square$$

Canonical Codensity Setting

Every codensity monad has a codensity setting:

$$
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F"] \arrow[d, equal] & \mathcal{C} \arrow[loop right, "\hat{F}"] \arrow[d, bend right, "N"'] \\
(\mathcal{C}_0^{op})^{op} \arrow[r, "\text{Ӳ}^{op}"] & {[\mathcal{C}_0, \mathrm{Set}]^{op}} \arrow[u, bend right, "R"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

conerve – totalization adjunction

$$NC = \mathcal{C}(C, F(-))$$

$$RC = \mathrm{Ran}_{\tilde{\text{Ӳ}}^{op}} F$$

$$\tilde{\text{Ӳ}} : \mathcal{C}_0^{op} \longrightarrow [\mathcal{C}_0, \mathrm{Set}]$$

contravariant Yoneda embedding

Leinster '13

$$\hat{F} \cong RN$$

BUT: This does not help to find a simple codensity presentation of a given monad.

Two Ingredients

① **Dual adjunctions** restricting to **simple** dualities

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "F"] \arrow[d, "S"', hook] & \mathcal{C} \arrow[loop right, "\hat{F}"] \arrow[d, bend right, "L"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{D}_0^{op} \arrow[r, "G^{op}"] & \mathcal{D}^{op} \arrow[u, bend right, "R"'] \end{tikzcd}$$

topology	sets,	topological sp.,	measurable sp.
vs.			
algebra	Boolean algs,	frames,	co-Boolean algs

② Compute **density** instead of **codensity**

$X \in \mathcal{D}$ "determined" by G-subobjects

$$\begin{tikzcd} GA \arrow[dr] \arrow[drr, bend left] & & \\ & GC \arrow[r] & X \\ GB \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

- $\mathbb{Q} \hookrightarrow \mathbb{R}$ dense
- $\{1\} \hookrightarrow \mathbf{Set}$ dense
- $\{a,b\}^* \hookrightarrow \mathbf{Mon}$ dense

formally dual, but "less natural"

e.g. $\{2\} \subseteq \mathbf{Stone}$

$\{S^1 \times S^1\} \subseteq \mathbf{AbGrp}(\mathbf{CHaus})$

well studied ☺

The Power of Category Theory

Clarity: separate generic from domain-specific parts

- $\widehat{\mathrm{Set}_f \hookrightarrow \mathrm{Set}} \cong \mathcal{U}$ is "completely categorical"
- $\widehat{\mathrm{Mark}_\omega \hookrightarrow \mathrm{Meas}} \cong$ Giry monad uses integral representation theorem

countably (discrete) Markov kernels

measurable spaces

probability distributions

Example: Neighbourhood Monad

$$\begin{tikzcd}
\{2\} \arrow[r, hook, "F"] & \mathrm{Set} \arrow[loop right, "2^{2^{-}}"] \arrow[d, bend right, "2^{-}"'] \arrow[d, phantom, "\dashv"] \\
\{1\}^{op} \arrow[u, "\wr"] \arrow[r, hook, "\iota^{op}"] & \mathrm{Set}^{op} \arrow[u, bend right, "2^{-}"']
\end{tikzcd}$$

discrete cats

$2^1 = 2$ ✓ $\qquad$ $\{1\} \overset{\iota}{\hookrightarrow} \mathrm{Set}$ dense ✓

$$\Longrightarrow \quad \hat{F} = 2^{2^{-}}$$

Example: Ultrafilter Monads

$$\begin{tikzcd}
\mathbf{Set}_f \arrow[r, hook, "I"] & \mathbf{Set} \arrow[loop right, "\mathfrak{U}"] \arrow[d, bend right, "\mathbf{Set}(-,2)"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{BA}_f^{op} \arrow[u, "E", "\wr"'] \arrow[r, hook, "J^{op}"] & \mathbf{BA}^{op} \arrow[u, bend right, "\mathbf{BA}(-,2)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Set}_f \arrow[r, hook, "\bar{I}"] & \mathbf{Top} \arrow[loop right, "\bar{\mathfrak{U}}"] \arrow[d, bend right, "\mathbf{Top}(-,2)"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{BA}_f^{op} \arrow[u, "E", "\wr"'] \arrow[r, hook, "J^{op}"] & \mathbf{BA}^{op} \arrow[u, bend right, "\mathbf{BA}(-,2)"']
\end{tikzcd}$$

ultrafilter of clopens

Birkhoff duality — $I \cdot E \cong \mathbf{BA}(J^{op}(-), 2)$ ✓

$\mathbf{BA}_f \xhookrightarrow{J} \mathbf{BA}$ dense ✓

$\Longrightarrow \quad \hat{I} \cong \mathfrak{U}$

Kennison & Gildenhuys '71

discrete

$\mathbf{Top}(X, 2) = $ clopens of X

$\circlearrowleft$ ✓ $\qquad J$ dense ✓

$\Longrightarrow \quad \hat{\bar{I}} = \bar{\mathfrak{U}}$

Sipoş '18

Example: Canonical Extension (Turning the Square)

$$
\begin{tikzcd}
\mathrm{BA}_f \arrow[r, hook, "J"] & \mathrm{BA} \arrow[loop right, "2^{\mathrm{Uf}(-)}"] \arrow[d, bend right, "\mathrm{BA}(-,2)"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{Set}_f^{op} \arrow[u, "E'", "\simeq"'] \arrow[r, hook, "I^{op}"] & \mathrm{Set}^{op} \arrow[u, bend right, "\mathrm{Set}(-,2)"']
\end{tikzcd}
$$

$$J \cdot E' \cong \mathrm{Set}(I^{op}(-), 2) \quad ✓$$

$\mathrm{Set}_f \overset{I}{\hookrightarrow} \mathrm{Set}$ dense ✓

$$\Longrightarrow \quad \hat{J} \cong 2^{\mathrm{Uf}(-)}$$

new

Filter Monads

Example: Filter Monads

$$\begin{tikzcd}
MSL_f \arrow[r, "U"] & Set \arrow[loop right, "\mathcal{F}"] \arrow[d, bend right, "2^-"'] \arrow[d, phantom, "\dashv"] \\
MSL_f^{op} \arrow[u, "2^-", "\wr"'] \arrow[r, hook, "J^{op}"] & MSL^{op} \arrow[u, bend right, "2^-"']
\end{tikzcd}
\Rightarrow \hat{U} \cong \mathcal{F}$$

$$\begin{tikzcd}
Rel_f = Kl_f(\mathcal{P}_f) \arrow[r, "U_{\mathcal{P}_f}"] & Set \arrow[loop right, "\mathcal{F}"] \arrow[d, bend right, "2^-"'] \arrow[d, phantom, "\dashv"] \\
Rel_f^{op} = Kl_f(\mathcal{P}_f)^{op} \arrow[u, "(-)^{op}", "\wr"'] \arrow[r, hook, "I^{op}_{\mathcal{P}_f}"] & EM(\mathcal{P}_f)^{op} = MSL^{op} \arrow[u, bend right, "2^-"']
\end{tikzcd}
\Rightarrow \hat{U}_{\mathcal{P}_f} \cong \mathcal{F}$$

<u>Variation:</u> topological filters

$$\begin{tikzcd}
Kl_f(\mathcal{P}_f) \arrow[r, "V_{\mathcal{P}_f}"] & Top \arrow[loop right, "\bar{\mathcal{F}}"] \arrow[d, bend right, "Top(-{,}\mathbb{S})"'] \arrow[d, phantom, "\dashv"] \\
Kl_f(\mathcal{P}_f)^{op} \arrow[u, "(-)^{op}", "\wr"'] \arrow[r, hook, "I^{op}_{\mathcal{P}_f}"] & MSL^{op} \arrow[u, bend right, "MSL(-{,}2)"']
\end{tikzcd}$$

$\bar{\mathcal{F}}X$ = filters of opensets of X

$MSL(X,2) \subseteq \mathbb{S}^{|M|}$ top. sp.

Sierpinski space

$V_{\mathcal{P}_f}X = \mathbb{S}^X$

$\Rightarrow \hat{V}_{\mathcal{P}_f} \cong \mathcal{F}$

Double Dualization Monads

Example: Double Dualization Monads

$$
\begin{tikzcd}
BA_f \arrow[r, "U"] & Set \arrow[loop right] \arrow[d, bend right, "2^-"'] \arrow[d, phantom, "\dashv"] \\
Set_f \arrow[u, "\simeq"] \arrow[r, hook, "I^{op}"] & Set^{op} \arrow[u, bend right, "2^-"']
\end{tikzcd}
\qquad
\begin{tikzcd}
MSL_f \arrow[r, hook, "I"] & MSL \arrow[loop right] \arrow[d, bend right, "2^-"'] \arrow[d, phantom, "\dashv"] \\
MSL_f^{op} \arrow[u, "\simeq"] \arrow[r, hook, "I^{op}"] & MSL^{op} \arrow[u, bend right, "2^-"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Vec_{fd} \arrow[r, hook, "I"] & Vec \arrow[loop right, "(-)^{**}"] \arrow[d, bend right, "(-)^*"'] \arrow[d, phantom, "\dashv"] \\
Vec_{fd}^{op} \arrow[u, "\simeq"] \arrow[r, hook, "I^{op}"] & Vec \arrow[u, bend right, "(-)^*"']
\end{tikzcd}
$$

$\Rightarrow \hat{U} \cong 2^{2^-}$

new

$\Rightarrow \hat{I} = MSL(MSL(-,2),2)$

Adámek, Sousa Adv. Math '21

$\Rightarrow \hat{I} \cong (-)^{**}$

Leinster TAC '13

Example: A more involved double dualization monad

torus

$$\begin{tikzcd}
\{S^1 \times S^1\} \arrow[r, "V"] & \mathrm{Ab} \arrow[loop right, "\mathfrak{B}"] \arrow[d, bend right, "\mathrm{Ab}(-,S^1)"'] \arrow[d, phantom, "\dashv"] \\
\{\mathbb{Z}\times\mathbb{Z}\}^{op} \arrow[u, "\wr"] \arrow[r, hook, "J^{op}"] & \mathrm{Ab}^{op} \arrow[u, bend right, "\mathrm{Ab}(-,S^1)"'] \arrow[r, "\cong", "\mathrm{Ab}(-,S^1)"'] & \mathrm{CHAb}
\end{tikzcd}$$

Pontryagin duality

$\mathfrak{B}X$ = Bohr compactification of X

$S^1 = \{x \in \mathbb{C} \mid |x| = 1\} \subseteq (\mathbb{C}_{\neq 0}, \cdot, 1)$

$V = \left(\{S_1 \times S_2\} \xhookrightarrow{I} \mathrm{CHAb} \xrightarrow{U} \mathrm{Ab}\right)$

$\circlearrowleft$: $S^1 \times S^1 \cong \mathrm{Ab}(\mathbb{Z}, S_1) \times \mathrm{Ab}(\mathbb{Z}, S_1) \cong \mathrm{Ab}(\mathbb{Z}\times\mathbb{Z}, S_1)$ ✓

J : $\{\mathbb{Z}\times\mathbb{Z}\} \xhookrightarrow{J} \mathrm{Ab}$ dense ✓

full subcat

$\Longrightarrow \quad \hat{V} \cong \mathfrak{B}$

new

The Monad of Isbell Duality

Example: Isbell Duality

$\sim$ is the dual adjunction between contra- & covariant presheaves

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "y"] \arrow[d, equal] & {[\mathcal{C}^{op}, Set]} \arrow[d, bend right, "\mathcal{O}"'] \arrow[d, phantom, "\dashv"] \\
(\mathcal{C}^{op})^{op} \arrow[r, "\tilde{y}^{op}"] & {[\mathcal{C}, Set]^{op}} \arrow[u, bend right, "Spec"']
\end{tikzcd}$$

$y : C \longmapsto \mathcal{C}(-, C)$ Yoneda

$\tilde{y} : C \longmapsto \mathcal{C}(C, -)$ embeddings

$\mathcal{O}X : (C \longmapsto Nat(X, \mathcal{C}(-, C)))$

$Spec\, X : (C \longmapsto Nat(X, \mathcal{C}(C, -)))$

$\circlearrowleft$ by Yoneda Lemma: $Spec(\tilde{y}(D))(C) = Nat(\mathcal{C}(D,-), \mathcal{C}(C,-))$

$\cong \mathcal{C}(D, C) = y(D)(C)$ ✓

$\tilde{y}$ is dense ✓

$\Longrightarrow \hat{y} \cong \underline{Spec \cdot \mathcal{O}}$ Kock '66, Di Liberti JPAA '20

Monads on top. Spaces

Example: Vietoris Monad

$\mathbb{V}X = \{ C \mid C \subseteq X,\ C \text{ closed}\}$ with "hit-&-miss" topology given by all

$U \subseteq X$ open

$$\Diamond U = \{ C \in \mathbb{V}X \mid C \cap U \neq \emptyset \} \qquad \text{"C hits U"}$$

$$\Box U = \{ C \in \mathbb{V}X \mid C \subseteq U \} \qquad \text{"C misses } X \setminus U\text{"}$$

$$\begin{tikzcd}
\mathrm{Rel}_f \cong \mathrm{Kl}_f(\mathcal{P}_f) \arrow[rr, "U_{\mathcal{P}_f}"] & & \mathrm{Stone} \arrow[loop right, "\mathbb{V}"] \arrow[d, bend right, "\mathrm{Stone}(-,2)"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{Rel}_f \cong \mathrm{Kl}_f(\mathcal{P}_f) \arrow[u, "(-)^{op}", "\wr"'] \arrow[rr, "I^{op}"] & & EM(\mathcal{P}_f) = \mathrm{JSL}^{op} \arrow[u, bend right, "\mathrm{JSL}(-,2)"']
\end{tikzcd}
\implies \hat{U}_{\mathcal{P}_f} \cong \mathbb{V}$$

Gehrke, Petrişan, Reggio MSCS'20

<u>Variants:</u>

Top with $\mathbb{V}_\downarrow$ lower Vietoris monad – new

Top with $\mathcal{S}$ sobrification monad – Sipoş Math. Slovak. '18

Probability Monads

Example: The Expectation Monad

$\mathcal{D}X = \{ \sum_{i=1}^{n} r_i x_i \mid r_i \in [0,1], x_i \in X, \sum_{i=1}^{n} r_i = 1 \}$ $\quad \mathbb{E}X = \{ \text{fin. add. prob. meas. } 2^X \to [0,1] \}$

finite stochastic matrices

$$\begin{tikzcd}
\mathrm{Mark}_f \cong \mathrm{Kl}_f(\mathcal{D}) \arrow[rr, "U_{\mathcal{D}}"] & & \mathrm{Set} \arrow[loop right, "\mathbb{E}"] \arrow[d, bend right, "{\mathrm{Set}(-,[0,1])}"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{EMod}_{fd}^{op} \arrow[u, "S"] \arrow[rr, "I^{op}"] & & \mathrm{EMod}^{op} \arrow[u, bend right, "{\mathrm{EMod}(-,[0,1]) \cong \mathrm{Stat}}"']
\end{tikzcd}$$

restricts $BA_f \simeq \mathrm{Set}_f$

Jacobs TCS '10

effect modules $\hat{=}$ "probabilistic vector spaces"

- ↻ ✓ $\qquad I$ dense ✓

Staton & Uijlen I&C '18

- $\mathbb{E}$ = monad of $\dashv$ follows from a <u>discrete integral repr. thm.</u>

Jacobs et al. I&C '16

$\Longrightarrow \widehat{U_{\mathcal{D}}} \cong \mathbb{E}$

Example: Other Probability Monads

Giry monad

$$\begin{tikzcd} \mathsf{Set}_c \arrow[r, "\mathcal{D}_c"] & \mathsf{Meas} \arrow[loop right, "\mathcal{G}"] \arrow[d, bend right, "{[0,1]^-}"'] \arrow[d, phantom, "\dashv"] \\ \mathsf{CABA}_c^{op} \arrow[u, "\simeq"] \arrow[r, hook, "I^{op}"] & \mathsf{EA}_\sigma^{op} \arrow[u, bend right, "{[0,1]^-}"'] \end{tikzcd}$$

σ-complete effect algebras

$$\Rightarrow \hat{\mathcal{D}}_c \cong \mathcal{G}$$

Radon monad

$$\begin{tikzcd} \mathsf{Set}_f \arrow[r, "\mathcal{D}"] & \mathsf{CHaus} \arrow[loop right, "\mathcal{R}"] \arrow[d, bend right, "{[0,1]^-}"'] \arrow[d, phantom, "\dashv"] \\ \mathsf{BA}_f^{op} \arrow[u, "\simeq"] \arrow[r, hook, "J^{op}"] & \mathsf{EA} \arrow[u, bend right, "{[0,1]^-}"'] \end{tikzcd}$$

effect algebras

$$\Rightarrow \hat{\mathcal{D}} \cong \mathcal{R}$$

Kantorovich monad

$$\begin{tikzcd} \mathsf{Set}_f \arrow[r, "\mathcal{D}_o"] & \mathsf{CMet} \arrow[loop right, "\mathcal{K}"] \arrow[d, bend right, "{[0,1]^-}"'] \arrow[d, phantom, "\dashv"] \\ \mathsf{BA}_f^{op} \arrow[u, "\simeq"] \arrow[r, hook, "J^{op}"] & \mathsf{EA} \arrow[u, bend right, "{[0,1]^-}"'] \end{tikzcd}$$

$$\Rightarrow \hat{\mathcal{D}}_o \cong \mathcal{K}$$

van Belle TAC'22

<u>non-trivial:</u>

$\mathcal{G}$ = "monad of $\dashv$" uses Daniell-Stone Integral Representation Theorem

Summary

* Codensity monads $\widehat{=}$ Duality + Density

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F"] \arrow[d, phantom, "\wr"] & \mathcal{C} \arrow[loop right, "\hat{F}"] \arrow[d, bend right, "L"'] \\
\mathcal{D}_0^{op} \arrow[r, "G^{op}"] & \mathcal{D}^{op} \arrow[u, bend right, "R"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

* all known instances arise in this way
* several new ones are obtained

Future Work

* use our framework to study codensity presentations of other monads
  e.g. variant of the Vietoris monad related to compact spaces and probabilistic power domains
  Jacobs & Furber LMCS'15

* develop our main theorem in Ross Street's formal theory of monads
  (i.e. monads in 2-categories)

* generalize to Street's monad extensions
  extend a monad along a 1-cell