Hybrid systems as coalgebras
Lyapunov morphisms for Zeno stability

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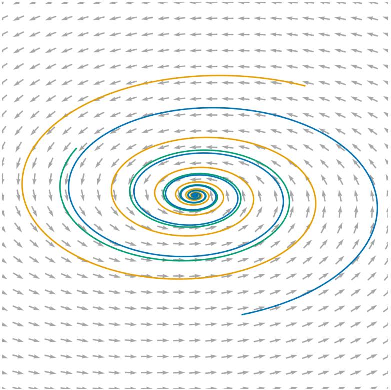
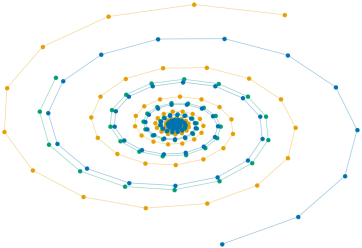
Stability

Continuous time:



Discrete time:



$$\dot{x} = -0.5x - 3y$$
$$\dot{y} = x - 0.5y$$

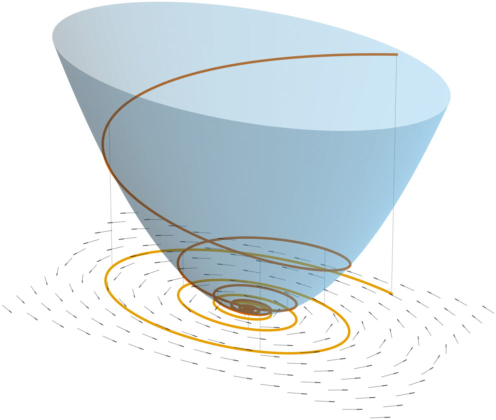
Lyapunov's Theorem

Theorem

Let $\dot{x} = f(x)$ *be a continuous time dynamical system, and* x^* *an equilibrium point. If* V *is a positive definite function w.r.t.* x^* *such that* $\frac{\partial V}{\partial x} \cdot f(x) \leq 0$*, then* x^* *is stable.*

Theorem

Let $x^+ = f(x)$ *be a discrete time dynamical system, and* x^* *an equilibrium point. If* V *is a positive definite function w.r.t.* x^* *such that* $V(f(x)) - V(x) \leq 0$*, then* x^* *is stable.*



$$V(x, y) = x^2 + 3y^2$$

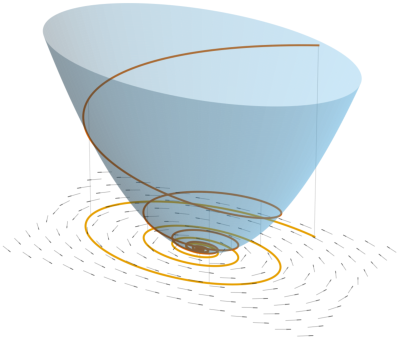
Categorical Lyapunov's Theorem

Theorem

Let $f\colon M \to \mathcal{F}(M)$ *be a coalgebra, and* $x^*\colon 1 \to M$ *an equilibrium. If* $V\colon M \to R$ *is a positive definite morphism w.r.t.* x^* *such that*

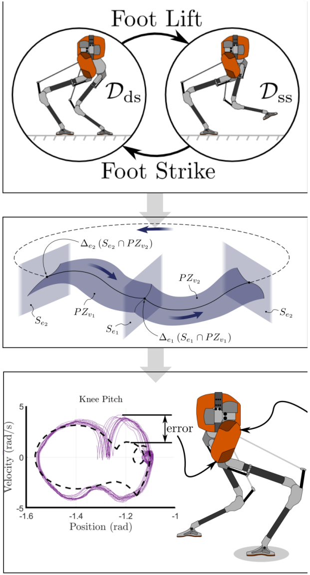
$$\begin{tikzcd} M \arrow[r, "V"] \arrow[d, "f"'] & R \arrow[d, "0"] \arrow[dl, Rightarrow] \\ \mathcal{F}(M) \arrow[r, "\mathcal{F}(V)"'] & \mathcal{F}(R) \end{tikzcd}$$

lax commutes, then x^* *is stable.*

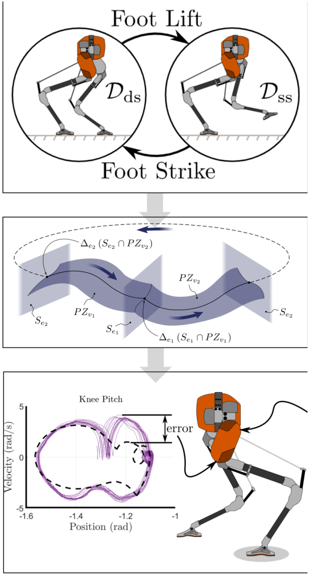


- $\mathcal{T}\colon \mathrm{Man} \to \mathrm{Man}$ coalgebras include vector fields, $\frac{\partial V}{\partial x} \leq 0$
- $\mathrm{id}\colon \mathrm{Top} \to \mathrm{Top}$ coalgebras are discrete systems, $V(f(x)) \leq V(x)$

Classical Hybrid Systems



Classical Hybrid Systems



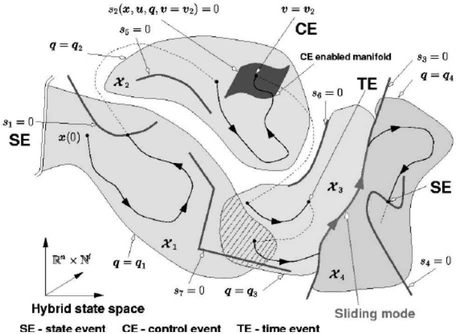
Definition

A **hybrid system** $H = (\Gamma, D, G, R, F)$:

- $\Gamma = (V, E)$: directed graph (modes, transitions)
- $D = \{D_v\}_{v \in V}$: manifolds
- $G = \{G_e\}_{e \in E}$: guards $G_e \subseteq D_{s(e)}$
- $R = \{R_e\}$: reset maps $R_e : G_e \to D_{t(e)}$
- $F = \{f_v\}$: vector fields $f_v : D_v \to \mathcal{T} D_v$

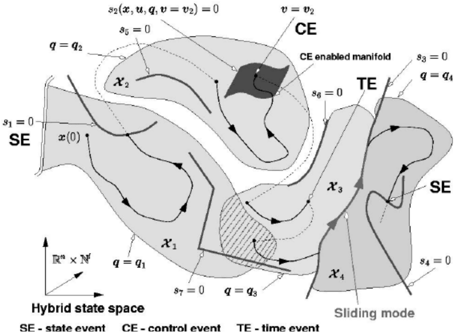
Goal: encode this as a coalgebra, then apply categorical Lyapunov theory.

Hybrid Time Trajectories



Hybrid time is two-dimensional: how long you have flowed (t), and how many jumps you have made (j).

Hybrid Time Trajectories



Hybrid time is two-dimensional: how long you have flowed (t), and how many jumps you have made (j).

Definition

An **execution** of H is a tuple $\chi = (\Lambda, \tau, \rho, c)$:

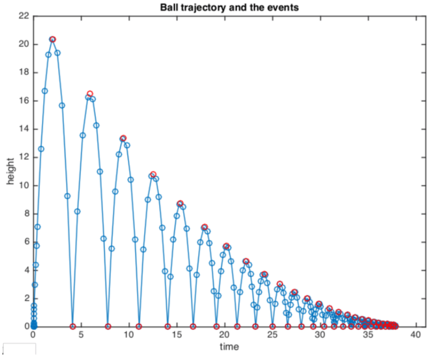
- $\Lambda \subseteq \mathbb{N}$ an indexing set, with jump times $\tau_j \leq \tau_{j+1}$ and intervals $I_j = [\tau_j, \tau_{j+1}]$
- $\rho \colon \Lambda \to V$ a mode sequence with $e_j := (\rho(j), \rho(j+1)) \in E$
- $c_j \colon I_j \to D_{\rho(j)}$ solution curves of $f_{\rho(j)}$

satisfying, at each jump time τ_{j+1}:

$$c_j(\tau_{j+1}) \in G_{e_j}$$
$$c_{j+1}(\tau_{j+1}) = R_{e_j}(c_j(\tau_{j+1}))$$

Flow in the current mode until you hit a guard, jump along the reset, repeat.

Zeno Stability

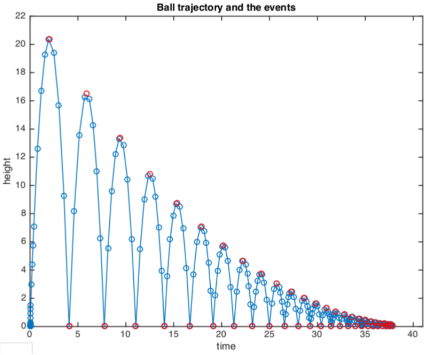


Each bounce scales velocity by $\lambda \in (0, 1)$, so bounce durations form a geometric series: infinitely many bounces before $\tau_\infty < \infty$.

An execution is **Zeno** if it makes infinitely many jumps in finite time.

- Everywhere in mechanical systems with collisions: bouncing balls, bipedal walking.
- Stalls simulation: the integrator must resolve infinitely many events.

Zeno Stability



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- Everywhere in mechanical systems with collisions: bouncing balls, bipedal walking.
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The ball at rest on the ground is the natural limit point z^*, but $f(z^*) = (0, -g)$, so classical Lyapunov theory doesn't apply.

The Category Chart

Definition

The category Chart blends Man and Set.

- **Objects**: pairs $\binom{S}{M}$, where M is a manifold and S is a set.
- **Morphisms** (charts): $\binom{f_d}{f_c} : \binom{S}{M} \rightrightarrows \binom{S'}{M'}$ consists of a smooth map $f_c \colon M \to M'$ and a function $f_d \colon S \times \underline{M} \to S'$.

Why: the discrete component f_d depends on the *continuous state* via $\underline{M}$ (the underlying set of M). Discrete transitions can depend on continuous position, as hybrid systems demand.

Products: $\binom{S}{M} \times \binom{S'}{M'} = \binom{S \times S'}{M \times M'}$. Terminal object: $\binom{1}{1}$.

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Chart $= \int (\mathit{Param} \colon \mathrm{Man} \to \mathrm{Cat})$

The Functor $\mathcal{H}\colon \mathsf{Chart} \to \mathsf{Chart}$ and $\mathcal{H}$-Coalgebras

On objects:

$$\mathcal{H}\begin{pmatrix} S \\ M \end{pmatrix} := \begin{pmatrix} \mathcal{P}(S \times \underline{M}) \\ \mathcal{T}M \end{pmatrix}$$

Blends:

- $\mathcal{T}\colon \mathsf{Man} \to \mathsf{Man}$ (tangent bundle) for continuous dynamics
- $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ (powerset) for discrete jumps

An **$\mathcal{H}$-coalgebra** $\binom{f_d}{f_c}\colon \binom{S}{M} \rightrightarrows \mathcal{H}\binom{S}{M}$ consists of:

- $f_c\colon M \to \mathcal{T}M$: a vector field on M
- $f_d\colon S \times \underline{M} \to \mathcal{P}(S \times \underline{M})$: a jump map
- $f_d(s, x) = \emptyset$: no jump available; the system flows.
- $f_d(s, x) \neq \emptyset$: any $(s', x') \in f_d(s, x)$ is a valid next state after a jump.

What are $\mathcal{H}$-coalgebras?

The jump map $f_d \colon S \times \underline{M} \to \mathcal{P}(S \times \underline{M})$ packages guard and reset into one morphism:

$$\text{guard} = \{(s,x) \mid f_d(s,x) \neq \emptyset\}, \qquad \text{reset} = \text{the values of } f_d.$$

$\mathcal{H}$-coalgebras interpolate between continuous and discrete dynamics:

Specialize	**Data remaining**	**System**
$S = 1,\ f_d \equiv \emptyset$	$f_c \colon M \to \mathcal{T}M$	vector field
$M = *$	$f_d \colon S \to \mathcal{P}(S)$	nondeterministic transition system
resets fix M, jumps anytime	$f_d(s,x) \ni (s',x)$	switching system
guards, resets, modes	everything	hybrid system

Hybrid Systems as $\mathcal{H}$-Coalgebras

Construction

Given $H = (\Gamma, D, G, R, F)$:

- $S = V, \quad M = \bigsqcup_{v \in V} D_v$
- $f_c = \bigsqcup_v f_v$ (assemble vector fields)
- $f_d(v, x) = \bigcup_{e \in E_v} \{(t(e), R_e(x))\}$ for $x \in G_e$, and $\emptyset$ otherwise

The $\mathcal{H}$-coalgebra framework also allows:

- S without a graph structure
- Nondeterministic jumps ($|f_d(s, x)| > 1$)
- Set-valued (non-manifold) guards

Time, solutions

A **solution curve** of $\vec{f}\colon M \to \mathcal{T}(M)$ is a smooth map $c\colon [0,a] \to M$ such that

$$\vec{g}(c(t)) = \dot{c}(t)$$

$$\begin{tikzcd}
\mathcal{T}([0,a]) \arrow[r, "dc"] & \mathcal{T}(M) \\
{[0,a]} \arrow[u, "\vec{1}"] \arrow[r, "c"'] & M \arrow[u, "\vec{f}"']
\end{tikzcd}$$

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\end{tikzcd}$$

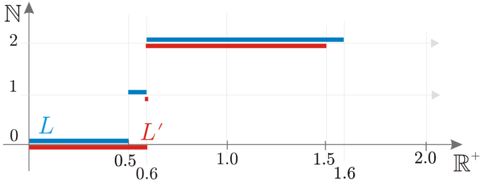
Definition

$\mathcal{C}$ has a collection of **time interval objects** *I each with a* **unit clock** *system $1_I\colon I \to \mathcal{F}(I)$. A* **solution curve** *is a map:*

$$\begin{tikzcd}
\mathcal{F}(I) \arrow[r, "\mathcal{F}(c)"] & \mathcal{F}(X) \\
I \arrow[u, "1_I"] \arrow[r, "c"'] & X \arrow[u, "f"']
\end{tikzcd}$$

Hybrid Time Domains

$$\mathbb{N} \hookleftarrow \Lambda \xrightarrow{\tau} \mathbb{R}^+$$



Hybrid time domain

$$\begin{pmatrix} \Lambda \\ \coprod_\Lambda [\tau_k, \tau_k + 1] \end{pmatrix}$$

Unit clock coalgebra:

$$1 = \begin{pmatrix} 1_d \\ 1_c \end{pmatrix} : \begin{pmatrix} \Lambda \\ \coprod_\Lambda I_j \end{pmatrix} \rightrightarrows \begin{pmatrix} \mathcal{P}(\Lambda \times \coprod_\Lambda \underline{I_j}) \\ \coprod_\Lambda \mathcal{T} I_j \end{pmatrix}$$

$$1_c : \coprod_\Lambda I_j \to \coprod_\Lambda \mathcal{T} I_j \qquad\qquad 1_c = \vec{1}$$

$$1_d : \Lambda \times \coprod_\Lambda \underline{I_j} \to \mathcal{P}(\Lambda \times \coprod_\Lambda \underline{I_j})$$

$$1_d(j, k, x) = \begin{cases} \{(j+1, j+1, \tau_{j+1})\} & j = k,\ t = \tau_{j+1} \\ \emptyset & \text{otherwise} \end{cases}$$

Zeno Equilibria

A **generalized element** $z^* \colon \binom{Z_d}{Z_c} \rightrightarrows \binom{S}{M}$ is **forward invariant** if every solution starting in its image stays there.

Definition

A **Zeno equilibrium** is a collection $\{z_v\}_{v \in V}$ with each $z_v \in G_e$ and $R_e(z_v) = z_{t(e)}$ for all outgoing edges e.

A Zeno equilibrium is a forward-invariant generalized element.

An execution is **Zeno** if $\Lambda = \mathbb{N}$ and

$$\tau_\infty := \lim_{j \to \infty} \tau_j < \infty.$$

Infinitely many jumps accumulate in finite time.

CLT for $\mathcal{H}$-Coalgebras

The abstract CLT (Papers I+II): $V : E \to R$ positive definite with

$$\begin{tikzcd}
E \arrow[r, "V"] \arrow[d, "f"'] & R \arrow[d, "\sigma"] \arrow[dl, Rightarrow] \\
\mathcal{F}E \arrow[r, "\mathcal{F}V"'] & \mathcal{F}R
\end{tikzcd}$$

implies stability.

Theorem (CLT for $\mathcal{H}$-Coalgebras)

Let $V = \left(\begin{smallmatrix} V_d \\ V_c \end{smallmatrix}\right)$ be positive definite w.r.t. z^*. Then z^* is stable if:

Flow condition:

$$\frac{dV_c}{dt} \cdot f_c(x) \leq \sigma_c(V_c(x))$$

Jump condition:

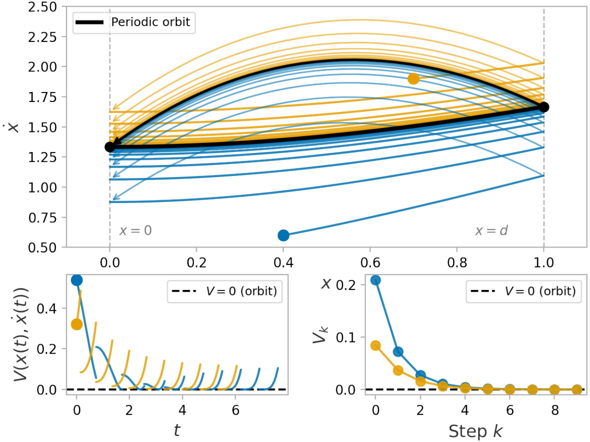
$$\big\{(V_d(s', x'),\ V_c(x')) \,\big|\, (s', x') \in f_d(s, x)\big\} \ \leq_H\ \sigma_d\big(V_d(s, x),\ V_c(x)\big)$$

Hybrid Periodic Orbits

With posetal object $\binom{1}{\mathbb{R}_{\geq 0}}$ (trivial discrete part), three known results fall out.

Corollary: Stability of Hybrid Periodic Orbits

Stability	$\sigma_c(a)$	$\sigma_d(a)$
Stable	0	$\{a\}$
Asymptotic	$-\alpha_3(a)$	$\{a - \beta(a)\}$
RES	$-(c/\varepsilon)\, a$	$\{\kappa_e\, a\}$



The Zeno Measurement Object

Definition

The **Zeno measurement object** is the $\mathcal{H}$-coalgebra $\sigma\colon \binom{\mathbb{R}_{\geq 0}}{\mathbb{R}_{\geq 0}} \rightrightarrows \mathcal{H}\binom{\mathbb{R}_{\geq 0}}{\mathbb{R}_{\geq 0}}$ given by

$$\sigma_c(r) = -c,$$

$$\sigma_d(a, r) = \begin{cases} \{(\lambda a,\ a)\} & r = 0, \\ \emptyset & r > 0, \end{cases}$$

for fixed $c > 0$ and $\lambda \in (0, 1)$.

- r is the continuous component: tracks time remaining to the next jump, decreases at rate c.
- Jumps only occur when $r = 0$.
- At each jump: $a \mapsto \lambda a$ (discrete part shrinks by λ); r resets to the previous a.
- Inter-jump durations form a geometric series, Zeno by construction.

Zeno Stability

Corollary (Stability of Zeno Equilibria)

Suppose $V_c, V_d : M \to \mathbb{R}_{\geq 0}$ satisfy for all $v \in V$, $e \in E$:

1. **Flow**: $\frac{dV_c}{dt} \cdot f_v(x) \leq -c$
2. **At each jump** ($x \in G_e$):

$$V_d(R_e(x)) \leq \lambda\, V_d(x)$$
$$V_c(R_e(x)) \leq V_d(x)$$

Then z^* is stable.

Theorem (Summability Bound)

Additionally suppose $\frac{dV_d}{dt} \cdot f_v(x) \leq 0$. Then for any execution with inter-jump durations Δt_k:

$$\sum_{k \in \Lambda} \Delta t_k \;\leq\; \frac{V_c\big(c_0(\tau_0)\big)}{c} + \frac{V_d\big(c_0(\tau_0)\big)}{c(1-\lambda)} \;<\; \infty.$$

If $\Lambda = \mathbb{N}$, the execution is Zeno.

Example: Bouncing Ball

Domain $D = \mathbb{R}_{\geq 0} \times \mathbb{R}$, guard $G = \{x_1 = 0,\ x_2 \leq 0\}$:

$$f_{\text{ball}}(x) = \begin{pmatrix} x_2 \\ -g \end{pmatrix},$$

$$R_{\text{ball}}(x) = \begin{pmatrix} 0 \\ -\lambda x_2 \end{pmatrix}, \lambda \in (0,1).$$

Let τ be time to next impact

$$\tau(x) = \frac{x_2 + \sqrt{x_2^2 + 2gx_1}}{g}$$

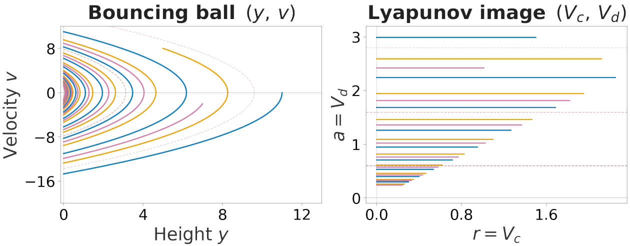
and υ be speed at next impact:

$$\upsilon(x) = \sqrt{x_2^2 + 2gx_1}.$$

Set $V_c = c\,\tau$ and $V_d = (2c/g)\,\upsilon$. All four conditions hold with equality. The summability bound gives:

$$\sum_{k\in\mathbb{N}} \Delta t_k \ \leq\ \tau(x_0) + \frac{2\,\upsilon(x_0)}{g(1-\lambda)}.$$

Bouncing ball $\xrightarrow{V}$ *target* σ_R



Simulation Morphisms and Transference of Stability

Definition

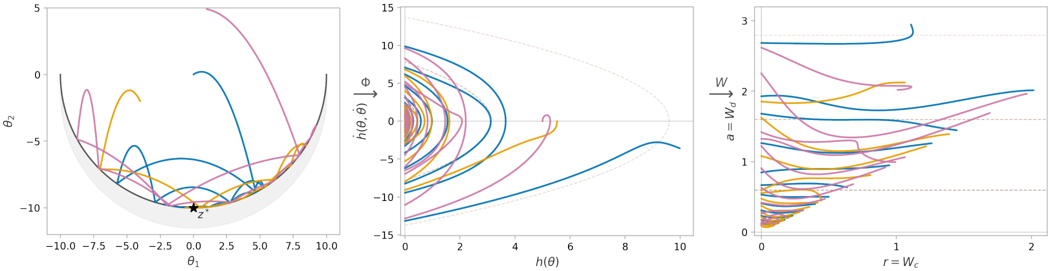
A chart $\Phi = \binom{\Phi_d}{\Phi_c} \colon \binom{S_E}{M_E} \rightrightarrows \binom{S_Y}{M_Y}$ is a **simulation morphism** from (E, f) to (Y, σ) if Φ satisfies the Lyapunov conditions for σ:

- **Continuous**: $\dfrac{d\Phi_c}{dt} \cdot f_c(x) \leq \sigma_c(\Phi_c(x))$
- **Discrete**: $\big\{(\Phi_d(s', x'), \, \Phi_c(x')) \;\big|\; (s', x') \in f_d(s, x)\big\} \leq \sigma_d(\Phi_d(s, x), \, \Phi_c(x))$

Theorem (Transference of Stability)

If $\Phi \colon (E, f) \to (Y, \sigma)$ is a simulation morphism and $V \colon (Y, \sigma) \to R$ is a Lyapunov morphism for y^*, then $W = V \circ \Phi$ certifies stability of $\Phi_c^{-1}(y^*(Z_c))$ in (E, f).

Application: Lagrangian Hybrid Systems



State $X = \mathcal{T}\Theta$, constraint $h\colon \Theta \to \mathbb{R}$, Euler-Lagrange vector field f_L, Newtonian impact reset with restitution $\lambda \in (0,1)$.

Zeno equilibria:
$Z_h = \{(\theta, \dot\theta) \mid h(\theta) = 0,\ \dot h(\theta, \dot\theta) = 0\}$.

Simulation map: $\Phi(\theta, \dot\theta) = (h(\theta),\ \dot h(\theta, \dot\theta))$ maps to the bouncing ball.

If $\ddot h(z^*) < 0$, set $\kappa = -\ddot h(z^*) > 0$ and compose with the bouncing ball Lyapunov functions (replacing g with κ):

$$W_c = c\,\tau_\kappa \circ \Phi, \qquad W_d = \frac{2c}{\kappa}\,\upsilon_\kappa \circ \Phi.$$

Every $z^* \in Z_h$ with $\ddot h(z^*) < 0$ is Zeno-stable.

Future Work

- **Control**: categorical CBFs and CLFs, control coalgebras, applications to ATMOS and other robotic systems.
- **Layered architectures**: compositional stability analysis of stacked hybrid controllers via simulation morphisms.
- **Collaborators**: Aaron Ames, Paulo Tabuada, Sebastian Mattenet, Pedro Roque, Yana Lishkova, Max de Sa
- **Mentees**: Teagan Abeling, Vivian Norum, Raina Ban, Deepak G, Juan Esteban Beron, Piash Mondol, Jacobo de Juan Millon

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