Quantum Finiteness Spaces

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The classical result

Theorem (Gelfand)

Let X be a compact hausdorff space. Let:

$$C(X) = \{f : X \to \mathbb{C} | f \text{ is continuous}\}$$

Then C is a contravariant functor:

$$C : CptHaus^{op} \longrightarrow CommC^{*}Alg$$

which is furthermore an equivalence of categories.

This equivalence restricts to an equivalence:

$$C : FinSet^{op} \longrightarrow CommC^{*}Alg_{fd}$$

Categorical quantum mechanics

- This provides an abstract characterization of the category of finite sets.
- But we would like to have something abstract enough so that it can be interpreted in monoidal categories, possibly with some additional structure.
- Why?
- **Categorical quantum mechanics** (Abramsky, Coecke)

Categorical quantum mechanics II

- QM is typically formulated in terms of operators on Hilbert spaces.
- A & C argue that it is really the tensor product on Hilbert spaces that is fundamental.
- They demonstrate this by encoding many quantum structures in monoidal categories.
- The basic structures studied are physical processes which can be composed due to the categorical structure and run in parallel using the monoidal structure.
- String diagrams, as in the work of Penrose and Joyal-Street, can be used to encode processes.
- String diagrams are also fundamental in linear logic via *proof nets*, a graph-theoretic syntax for specifying proofs.

Categorical quantum mechanics III

- Adding some simple axioms to monoidal categories allows one to reconstruct the category of finite-dimensional Hilbert spaces. (Heunen-Kornell)
- The fact that in a general monoidal category we typically don't have maps $X \to X \otimes X$ or $X \to I$ correspond to the no-cloning and no-deleting theorems of QM.
- From Coecke-Pavlovic-Vicary: "...we rely on the distinct ability to clone and delete classical data as compared to quantum data..."
- So sitting inside our monoidal category of quantum processes is a "classical" category in which we can copy and delete data. This will correspond to a canonical category equivalent to the category of finite sets inside the category of Hilbert spaces.
- These are the *special dagger commutative Frobenius algebras*.

Frobenius Algebras

Definition

A **Frobenius algebra** is a vector space A equipped with an associative multiplication, unit, coassociative comultiplication, and counit

$$\mu : A \otimes A \to A\,, \quad \eta : k \to A$$

$$\delta : A \to A \otimes A\,, \quad \epsilon : A \to k$$

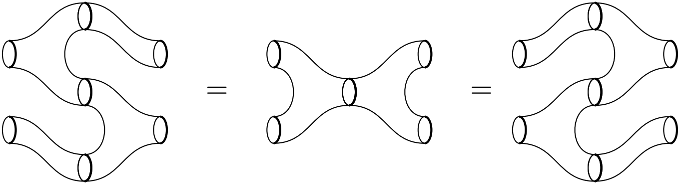
respectively, such that the Frobenius relation below holds

$$(id \otimes \mu) \circ (\delta \otimes id) = \delta \circ \mu = (\mu \otimes id) \circ (id \otimes \delta)$$

We can generalise this to a **Frobenius object** in a monoidal category.
A Frobenius algebra is a Frobenius object in $(Vect_k, \otimes, k, \sigma)$.

Frobenius Algebras II

Frobenius relation $(id \otimes \mu) \circ (\delta \otimes id) = \delta \circ \mu = (\mu \otimes id) \circ (id \otimes \delta)$



Dagger Categories and Dagger Frobenius Algebras

- A monoidal category is *dagger monoidal* if equipped with an involutive functor $(-)^{\dagger}$ which is the identity on objects and commutes with all the monoidal structure, e.g. $\alpha^{\dagger} = \alpha^{-1}, (f \otimes g)^{\dagger} = f^{\dagger} \otimes g^{\dagger}$, etc.

- A Frobenius algebra is a *dagger Frobenius algebra* if $\delta^{\dagger} = \mu$ and $\eta^{\dagger} = \varepsilon$.

- A dagger Frobenius algebra is *dagger commutative* if $c^{\dagger} = c^{-1}$.

Key Example Of A Dagger Commutative Frobenius Algebra

- The category FDHilb of finite-dimensional Hilbert spaces is a dagger category under the adjoint operation.

- Let H be a finite-dimensional Hilbert space, and $\{v_i\}$ be an orthonormal basis. Define:

$$\delta(v_i) = v_i \otimes v_i \qquad \epsilon(v_i) = 1$$

  This determines a dagger commutative Frobenius algebra structure on H which is furthermore *special* in the sense that

$$\delta \circ \mu = id_{\mathrm{H}}$$

Key Example Of A Dagger Commutative Frobenius Algebra II

Conversely, given a special dagger commutative Frobenius algebra structure on a Hilbert space H, define the *copyable elements* to be those elements such that

$$\delta(v) = v \otimes v \qquad \epsilon(v) = 1$$

Quite generally, given a coalgebra in a category of vector spaces, the copyable elements will be linearly independent.

Here we can say more than that.

Lemma

In the above situation, the copyable elements form an orthonormal basis for H.

The Theorem

For morphisms between Frobenius algebras, one takes those maps which preserve the coalgebra structure. If we require maps preserve both algebra and coalgebra, the only maps in the category will be isomorphisms. Since our maps preserve coalgebra structure, copyable elements are taken to copyable elements, and furthermore:

Theorem (Coecke-Pavlovic-Vicary)

These two constructions are inverse to each other. Furthermore the constructions can be extended to an equivalence of categories between the category of finite sets and the category of special dagger commutative Frobenius algebras (in FDHilb*).*

Everything's Finite

The proof depends crucially on the spectral theorem for finite-dimensional C*-algebras.

If A is an algebra in the category of Hilbert spaces and $a \in A$. Define a map $R_a \colon A \to A$ by $R_a(b) = ba$. Then consider $R_a^\dagger$. For a general algebra, there need not be an element a^* such that $R_a^\dagger = R_{a^*}$. But in our setting, it does exist. In fact it's given by a specific formula.

Theorem (Coecke-Pavlovic-Vicary)

If A is a commutative $\dagger$-Frobenius algebra, then the element a^* exists uniquely. Furthermore, the operation $(-)^*$ turns A into a C-algebra.*

Everything's Finite II

This allows us to use the structure theorem for finite-dimensional C*-algebras.

Everything about this construction is finite. We're working in the category FDHilb, and the equivalence is with the category of finite sets. But this is forced.

Lemma

If a vector space V has the structure of a Frobenius algebra, it is necessarily finite-dimensional.

If we want infinite dimensions to come into play, we need new structures.

- We'll replace finite sets with *finiteness spaces* (Ehrhard).
- We'll replace discrete vector spaces with Lefschetz spaces.
- We'll replace dagger commutative Frobenius algebras with dagger commutative linear monoids (Srinivasan).

(Multiplicative) Linear Logic (J.-Y. Girard)

We'll advance the idea that (multiplicative) linear logic could act as a type theory for quantum processes. The categorical structure of MLL is the $*$-*autonomous category*.

Definition (Barr)

Let $\mathcal{C}$ be a symmetric monoidal closed category and $\perp$ an object of $\mathcal{C}$. Then $\perp$ is a *dualizing object* if the canonical map:

$$\rho \colon V \to (V \multimap \perp) \multimap \perp = V^{\perp\perp}$$

A symmetric monoidal closed category with a dualizing object is a $*$-autonomous category.

The involutive negation allows us to define a second monoidal structure via de Morgan duality:

$$A \oplus B = (A^{\perp} \otimes B^{\perp})^{\perp}$$

Another approach is due to Cockett and Seely. Take the $\otimes$ and $\oplus$ as primitive.
An *LDC* is a category with two monoidal structures, $\otimes, \top$ and $\oplus, \perp$ and natural transformations

$$\partial \colon A \otimes (B \oplus C) \to (A \otimes B) \oplus C, etc$$

satisfying equations. One can reintroduce negation by maps

$$\rho \colon A \otimes A^{\perp} \to \perp \qquad\qquad \sigma \top \to A^{\perp} \oplus A$$

satisfying equations. This presentation is equvalent to the theory of $*$-autonomous categories.

Another approach (Abramsky & Heunen)

In their paper, *H*-algebras and nonunital Frobenius algebras: first steps in infinite-dimensional categorical quantum mechanics*,

Lemma

A Frobenius algebra in the category of Hilbert spaces is unital if and only if it is finite-dimensional.

They explore different axioms one can add to the definition of non-unital Frobenius algebras, all of which make sense in arbitrary dagger monoidal categories.

The Slogan

Working with finiteness spaces forces certain sums which would a priori be infinite to become finite, and hence well-defined, without resorting to limit structure. This will lead to a great many applications.

Ehrhard's finiteness spaces I

Let X be a set and let $\mathcal{U}$ be a set of subsets of X, i.e., $\mathcal{U} \subseteq \mathcal{P}(X)$. Define $\mathcal{U}^\perp$ by:

$$\mathcal{U}^\perp = \{u' \subseteq X \mid \text{the set } u' \cap u \text{ is finite for all } u \in \mathcal{U}\}$$

Lemma

- $\mathcal{U} \subseteq \mathcal{U}^{\perp\perp}$
- $\mathcal{U} \subseteq \mathcal{V} \Rightarrow \mathcal{V}^\perp \subseteq \mathcal{U}^\perp$
- $\mathcal{U}^{\perp\perp\perp} = \mathcal{U}^\perp$

A *finiteness space* is a pair $\mathbb{X} = (X, \mathcal{U})$ with X a set and $\mathcal{U} \subseteq \mathcal{P}(X)$ such that $\mathcal{U}^{\perp\perp} = \mathcal{U}$. We will sometimes denote X by $|\mathbb{X}|$ and $\mathcal{U}$ by $\mathcal{F}(\mathbb{X})$. The elements of $\mathcal{U}$ are called *finitary* subsets. The elements of $\mathcal{U}^\perp$ are called *cofinitary* subsets.

Finiteness spaces II: Morphisms

- A *morphism* of finiteness spaces $R\colon \mathbb{X} \to \mathbb{Y}$ is a relation $R\colon |\mathbb{X}| \to |\mathbb{Y}|$ such that the following two conditions hold:
  (1) For all $u \in \mathcal{F}(\mathbb{X})$, we have $uR \in \mathcal{F}(\mathbb{Y})$, where $uR = \{y \in |\mathbb{Y}| \mid \exists x \in u, xRy\}$.
  (2) For all $v' \in \mathcal{F}(\mathbb{Y})^{\perp}$, we have $Rv' \in \mathcal{F}(\mathbb{X})^{\perp}$.

Composition is relational and it is straightforward to verify that this is a category. We denote it FinRel. It's a $*$-autonomous category.

Finiteness spaces III: Other choices of morphism

Definition

We define the category FinF*. Objects are finiteness spaces and a morphism* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ *is a function satisfying the same conditions as above. We define the category* FinPf*. Objects are finiteness spaces and a morphism* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ *is a partial function satisfying the same conditions as above.*

Proposition

The category FinPf *is a symmetric monoidal closed, complete and cocomplete category.*

Linearizing finiteness spaces

Let R be a field and $\mathbb{X} = (X, \mathcal{U})$ a finiteness space. Ehrhard defined the R-module $R\langle\mathbb{X}\rangle$ as the set

$$R\langle\mathbb{X}\rangle = \{f : X \to R \mid supp(f) \in \mathcal{U}\}$$

together with pointwise addition, where:

$$supp(f) = \{x \in X | f(x) \neq 0\}.$$

This is an R-module and furthermore $R\langle\mathbb{X}\rangle$ can be given the structure of a reflexive Lefschetz space (Ehrhard). This construction respects the $*$-autonomous structure.

Lemma (Ehrhard)

There is a linear isomorphism between $R\langle\mathbb{X} \multimap \mathbb{Y}\rangle$ and the R-module of linear continuous maps from $R\langle\mathbb{X}\rangle$ to $R\langle\mathbb{Y}\rangle$.

Linearizing II

Theorem

If $(\mathbb{M}, \mu\colon \mathbb{M} \otimes \mathbb{M} \to \mathbb{M}, \eta\colon I \to \mathbb{M})$ *is a finiteness monoid and* R *a ring (not necessarily commutative, but with unit), then* $R\langle\mathbb{M}\rangle$ *canonically has the structure of a ring.*
The multiplication in $R\langle\mathbb{M}\rangle$ *is given by*

$$(f \cdot g)(m) = \sum_{(m_1,m_2)\in X_m(f,g)} f(m_1) \cdot g(m_2).$$

where:
$X_m(f,g) = \{(m_1, m_2) \mid \mu(m_1, m_2) = m,\ f(m_1) \neq 0,\ g(m_2) \neq 0\}$

We can replace ring with semiring or even ordered ring or ordered semiring and the above still works.

The sum is finite.

Why is the set $X_m(f, g)$ finite?

This set is exactly

$$\underbrace{(\text{supp}(f) \times \text{supp}(g))}_{\in \mathcal{W}} \cap \underbrace{\mu^{-1}(m)}_{\in \mathcal{W}^{\perp}}$$

Recall that μ is the multiplication. $\mathcal{W}$ is the finiteness space structure for $\mathbb{M} \otimes \mathbb{M}$.

Topological groupoids

These can be defined with various levels of generality. We'll follow A. Sims, *Hausdorff étale groupoids and their C^*-algebras*

- A *topological groupoid* is a groupoid $\mathcal{G}$ in the category of locally compact hausdorff spaces and continuous maps.
- A topological groupoid is *étale* if its domain map is a local homeomorphism. (This implies the range map and multiplication are as well.)

Associating algebras to étale groupoids

Theorem

Let $\mathcal{G}$ be a second countable étale groupoid. Let

$$\mathcal{C}_c(\mathcal{G}) = \{f : \mathcal{G} \to \mathbb{C} | supp(f) \ \ \text{is compact}\}$$

Define $f \star g : \mathcal{G} \to \mathbb{C}$ by

$$f \star g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$$

Then $\mathcal{C}_c(\mathcal{G})$ is a $*$-algebra with above multiplication and $f^*(\gamma) = \overline{f(\gamma^{-1})}$.

The key point is that this sum is finite. There is a purely topological argument to support this. For us, the finiteness of the sum follows from properties of finiteness spaces.

Topological spaces as finiteness spaces?

It makes sense whether to ask if there is a class of sufficiently nice topological spaces X such that $(X, \mathcal{U})$ is a finiteness space where $\mathcal{U}$ is the set of relatively compact subsets and $\mathcal{U}^{\perp}$ is the set of discrete, closed subspaces. (A subspace is *relatively compact* if its closure is compact in X.)

For general topological spaces, this is certainly false. Furthermore, it is not sufficient to assume the space is locally compact, hausdorff.

Topological spaces as finiteness spaces I

The following is the work of Joey Beauvais-Feisthauer, Ian Dewan & Blair Drummond.

Theorem

The smallest uncountable ordinal ω_1, with the order topology, is locally compact and Hausdorff but not a finiteness space under the above structure.

But a smaller class of spaces does work.

Definition

- *X is σ-compact if it can be covered by a countable family of compact subsets.*
- *X is σ-locally compact if it is both σ-compact and locally compact.*

Topological spaces as finiteness spaces II

Theorem (B-F,D,D)

- *Let X be a σ-locally compact hausdorff space. Then it is a finiteness space.*
- *The converse is false. Let X be an uncountable discrete space. Then X is locally compact and hausdorff, but not σ-compact. But X is a finiteness space.*

Nonetheless, the class of σ-locally compact Hausdorff spaces is quite large, e.g. it contains manifolds and every CW-complex with countably many cells.

Some of our étale groupoids have underlying spaces which are σ-locally compact Hausdorff. We will use this fact to give a new approach to constructing algebras for them.

The following is due to Kumjian, Pask, Raeburn and Renault. We'll call this the *KPRR-groupoid*. See the paper:

Graphs, groupoids and Cuntz-Krieger algebras, by the above authors. They show that the C*-algebras of this form are of fundamental importance. One of their theorems, stated somewhat imprecisely.

Theorem

The C-algebra attached to a graph of the above form is the universal C*-algebra generated by (possibly infinite) families of partial isometries satisfying Cuntz-Krieger relations determined by the graph.*

A finiteness topological groupoid II

Let $G = (V, E)$ be a directed graph with V countable. We'll also assume G is row-finite, i.e. for all vertices v, $d^{-1}(v)$ is finite. Let $P(G)$ be the set of all infinite paths and $F(G)$ be the set of all finite paths. $P(G)$ can be seen as a subspace:

$$P(G) \subseteq \prod_{i=1}^{\infty} E \quad \text{with } E \text{ topologized discretely}$$

The topology can be described as follows. If $\alpha \in F(G)$, let

$$Z(\alpha) = \{x \in P(G) | x = \alpha y, \quad \text{with } y \in P(G)\}$$

Theorem (KPRR)

The sets $\{Z(\alpha) | \alpha \in F(G)\}$ form a basis for the topology on $P(G)$. The resulting topology is locally compact and totally disconnected.

A finiteness topological groupoid III

$P(G)$ is the object part of an étale groupoid.

Definition

Suppose $x, y \in P(G)$. We say that x and y are *shift equivalent with lag* $k \in \mathbb{Z}$ if there exists $N \in \mathbb{N}$ such that $x_i = y_{i+k}$ for all $i > N$. We write $x \sim_k y$.

Lemma

We have $x \sim_0 x$ and $x \sim_k y \Rightarrow y \sim_{-k} x$ and $x \sim_k y, y \sim_l z \Rightarrow x \sim_{k+l} z$.

Define

$$\mathcal{G} = \{(x, k, y) \in P(G) \times \mathbb{Z} \times P(G) | x \sim_k y\}$$

A finiteness topological groupoid IV

Define a multiplication $\mu \colon \mathcal{G}^2 \to \mathcal{G}$

$$\mu((x,k,y_1)(y_2,l,z)) \mapsto$$
$$\begin{cases} \text{undefined} & \text{if } y_1 \neq y_2 \\ (x, k+l, z) & \text{if } y_1 = y_2 \end{cases}$$

with inverse given by $i(x,k,y) = (y,-k,x)$

Theorem

Let G be a row-finite directed graph. The sets

$$\{Z(\alpha,\beta) | \alpha,\beta \in F(G) \text{ and } r(\alpha) = r(\beta)\}$$

form a basis for a locally compact Hausdorff topology on $\mathcal{G}$. With this topology, $\mathcal{G}$ is a second countable, locally compact and σ-locally compact étale groupoid.

It works

Theorem (R. Blute, J. Beauvais-Feisthauer, I. Dewan, B. Drummond, P.-A. Jacqmin)

Let K be a commutative ring. $\mathcal{G}$ is a finiteness space with the relatively compact structure. The multiplication of the groupoid makes this a finiteness monoid. So linearization gives us a K-algebra. If K has a $*$-operation, then the result is a $*$-algebra.

But at the moment, our construction is discrete. In the groupoid approach to C*-algebras, after building the $*$-algebra, one defines a norm and then completes with respect to the norm to obtain a C*-algebra. Can we add topology in our construction? Yes, but this is quite different than the usual topology of functional analysis.

Lefschetz topology

Lefschetz introduced this notion of topology with the intent of having infinite-dimensional vector spaces which are isomorphic to their second dual.

Definition

A vector space is a *Lefschetz space* if equipped with a T_0-topology such that

- The vector operations are continuous, i.e. it is a topological vector space. (We'll assume that the base field is discrete.)
- $0 \in V$ has a neighborhood basis of open linear subspaces.

The category of Lefschetz spaces and continuous linear maps will be denoted Lef.

Lefschetz topology II

Lemma (Barr)

Lef *is symmetric, monoidal closed.*

Lemma (Lefschetz)

The embedding $\rho \colon V \to V^{**}$ *is a bijection for all Lefschetz spaces.*

Definition

Let RLef *be the full subcategory of reflexive objects, i.e. those objects for which* ρ *is an isomorphism.*

Theorem (Barr)

RLef *is a* $*$*-autonomous category. In fact,* RLef *is a reflective subcategory of* Lef *with reflection given by* $(-)^{**}$.

As observed by Ehrhard, there is a topology one can place on the linearizations of finiteness spaces.

Definition

Let $(X, \mathcal{U})$ be a finiteness space. Let $u' \in \mathcal{U}^{\perp}$. Let

$$V_{u'} = \{f \in R\langle X\rangle | supp(f) \cap u' = \emptyset\}$$

This determines a neighborhood basis at the point $0 \in R\langle X\rangle$. The resulting topology is a Lefschetz topology.

Lefschetz topology IV

Here are some properties, as observed by Ehrhard, for a finiteness space $(X, \mathcal{U})$.

Lemma

- *If $\mathcal{U}$ consists of just the finite subsets, then $R\langle X\rangle$ gets the discrete topology.*
- *If $\mathcal{U} = \mathcal{P}(X)$, then $R\langle X\rangle = R^X$ with the product topology*

This topology is very different from the C^* one discussed earlier.

Dagger LDCs

The difficult thing about dagger LDCs is that the dagger can no longer be the identity on objects.

Then we need a *dagger functor* $(-)^\dagger$ which is a monoidal contravariant equivalence, so I have isomorphisms:

$$(A \otimes B)^\dagger \cong A^\dagger \oplus B^\dagger \qquad A \cong A^{\dagger\dagger} \qquad \top \cong \bot^\dagger$$

satisfying equations.

Dagger LDCs II

In many examples, especially those arising in physics, we'll be working over the complex numbers, and so also need a covariant conjugation functor $\overline{(-)}$ (which can also be taken to be the identity in many examples). This comes with isos:

$$id \cong \overline{\overline{(-)}} \qquad \overline{A \otimes B} \cong \overline{A} \otimes \overline{B} \qquad etc$$

All of this structure must satisfy some equations, all of which can be found in Priyaa Srinivasan's thesis.

Dagger Linear Monoids

The LDC analog of a Frobenius algebra is a *linear monoid*, notion due to Priyaa.

Definition

Let C be an LDC. A *linear monoid* consists of

- A tensor monoid $(m\colon A \otimes A \to A, u\colon \top \to A)$
- A par comonoid $(\delta\colon B \to B \oplus B, \epsilon\colon B \to \bot)$
- Actions $\rho\colon A \otimes B \to B, A \to B \oplus A$.

These must satisfy some (by which I mean many, many) equations.

A *dagger linear monoid* is defined similarly as before.

Conjecture

The copyable elements of a dagger linear monoid in the category of reflexive Lefschetz spaces is a finiteness space and furthermore there is an equivalence of categories between an appropriate category of dagger linear monoids in that category and the category of finiteness spaces and functions.

This turns out to be false. The project was stuck until Durgesh Kumar made a great deal of progress on the question in his M.Sc. Thesis.

The trouble with counits

The problem lies with requiring a counit for the comultiplication:

Lemma (Kumar)

If we have a dagger linear monoid including a counit, then the topology on the underlying Lefschetz space must be discrete.

So we'll drop the counit and (hence the unit) from the definition of dagger linear monoid.

However, as shown by Kumar, given a finiteness space, one can construct a counitless dagger linear monoid, by roughly the same construction as in Coecke-Pavlovic-Vicary, but now there's topology and two tensors ($\otimes$ and $\oplus$) to worry about.

Finding a finiteness space

Constructing a finiteness space from a family of linearly independent vectors in a Lefschetz space is much more complicated. It requires the following, called *The Scalar Invariance Theorem*

Theorem (Kumar)

Let V be a complete Lefschetz space with basis[a] $\{v_i\}_{i\in I}$. *Let* $J \subseteq I$. *If*

$$\sum_{j\in J} \alpha_j v_j$$

converges for some choice of non-zero coefficients, then

$$\sum_{j\in J} \beta_j v_j$$

for any choice of coefficients.

[a] The proper notion of basis is topological.

Finding a finiteness space II

Definition

Let V be a reflexive Lefschetz space with basis $\mathcal{B} = \{v_i\}_{i \in I}$. Define

$$\mathcal{U} = \{S \subseteq \mathcal{B} | \sum_{s \in S} \alpha_s v_s\}$$

where the α_s's are some (hence any) choice of nonzero coefficients.

Theorem (Kumar)

The above determines a finiteness space structure on $\mathcal{B}$.

The Adjunction

We now have a construction in both directions:

Finiteness Spaces $\longrightarrow$ Lefschetz spaces (Ehrhard)

Lefschetz spaces $\longrightarrow$ Finiteness Spaces (Kumar)

Theorem (Kumar)

These constructions are functorial and form an adjunction which is not an equivalence.

The problem is that while the copyable elements are still linearly independent, they do not in general form a basis.

Thanks

Thanks for listening.