Deriving semantics from pragmatics

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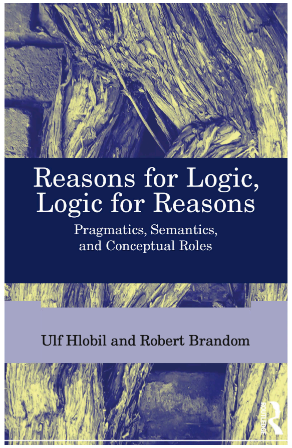
9/3/26

Background (1/6)



(2025)

Robert Brandom

Ulf Hlobil

Project Timeline

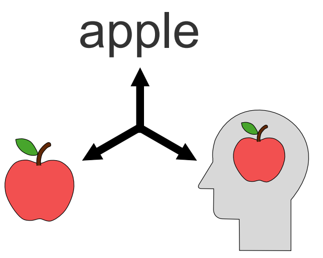
- 2024: Formal pragmatics $\longrightarrow$ semantics construction for *propositional* logics

Background (2/6): semantics

Competing strategies for understanding the *meanings* of our **ordinary** language:

The semantic attitude

Bits of language have meaning by *referring* to the world.



One possibility:

- particular nouns pick out **objects** (from some set of 'real-world objects')
- adjectives pick out **subsets** of objects (i.e. predicates)
- sentences pick out **subsets of possible worlds** (the worlds in which they're true)
- and picks out the intersection of such subsets

Background (3/6): semantic analysis example

$$
\begin{gathered}
[\![\text{Amy is poor and honest}]\!] \\
[\![\text{Amy is poor}]\!][\![\text{and}]\!][\![\text{Amy is honest}]\!] \\
\underbrace{[\![\text{Poor}]\!]}_{\text{Predicate}}(\underbrace{[\![\text{Amy}]\!]}_{\text{Object}}) \quad \underbrace{\cap}_{\text{Connective}} \quad \underbrace{[\![\text{Honest}]\!]}_{\text{Predicate}}(\underbrace{[\![\text{Amy}]\!]}_{\text{Object}})
\end{gathered}
$$

$$\text{As conjunctions: } [\![\text{and}]\!] = [\![\text{but}]\!] = \cap$$

$$[\![\text{Amy is poor and honest}]\!] = [\![\text{Amy is poor but honest}]\!]$$

$$[\![\text{Amy is poor and honest}]\!] \vDash [\![\phi]\!] \quad \text{iff} \quad [\![\text{Amy is poor but honest}]\!] \vDash [\![\phi]\!]$$

Upshot: semantic attitude accounts for compositionality of natural language

Background (4/6): pragmatics

The pragmatic attitude

To say something is to *do* something, a speech act.

The meanings of speech acts are to be read off of their consequences.

- E.g. “Court is in session” not a description of the court
- E.g. propriety of a chess move not justified by reference.
  - Some particular knight movement allowed *because* we have agreed on the valid moves.

A notion of consequence from pragmatic raw materials (Restall 2005)

- Assume we have at least two specific kinds of speech acts: *assertion* and *denial*.
- Assume a normative status ‘impropriety’, e.g. $\text{Improper}(A^+)$ means it’s improper to assert A.
- $\text{Improper}(A^+,\ B^-) \ \equiv\ A \vdash B \ \equiv\ B \text{ follows from } A \ \equiv\ A \text{ is a reason for } B$

Background (5/6): semantics versus pragmatics

Competing strategies for understanding the *meanings* of our **ordinary** language:

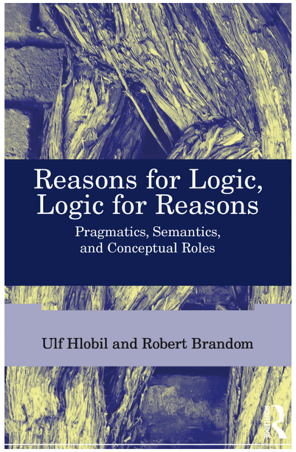
The semantic attitude	The pragmatic attitude
Language has meaning by *referring* to the world.	To say something is to *do* something (speech act). Meanings are to be read off of the consequences.

$$\text{When all goes well:} \quad [\![A]\!] \vDash [\![B]\!] \quad \text{iff} \quad A \vdash B$$

$$\text{However:} \quad [\![\text{Amy is poor but honest}]\!] \nvDash [\![\text{Wealth+honesty are related}]\!]$$

$$\text{... yet:} \quad [\![\text{Amy is poor but honest}]\!] \vdash [\![\text{Wealth+honesty are related}]\!]$$

Background (6/6): project timeline

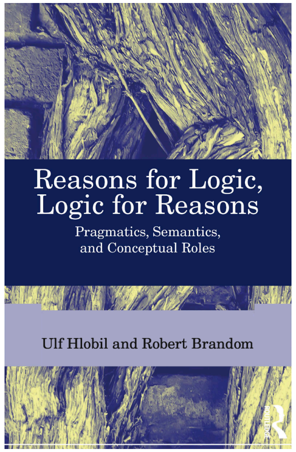


- 2024: Formal pragmatics $\longrightarrow$ semantics construction for *propositional* logics

Problems

1. Concise presentation of the technical formalism, showing design decisions as 'natural'
2. Generalization to syntax with subsentential structure (e.g. predicate logic)

Background (6/6): project timeline



Project Timeline

- 2024: Formal pragmatics $\longrightarrow$ semantics construction for *propositional* logics
- 2025: Described as the η of an adjunction involving simple, commonplace categories.

- 2026: Generalized internally to any $\mathcal{E}$ (some fixed topos with a natural numbers object)
- 2026: Instantiate $\mathcal{E} \mapsto \mathsf{Nom}$, use particularities of nominal sets to address predicate logic.

Technical Outline

1. Internal (pointed, Girard) quantales
2. The ‘free sequent-set’ functor
3. Implication frames + the logical completion
4. Interpreting MALL and classical logic in a frame
5. **Work in progress**: Nominal sets to model subsentential structure

Quantales internal to a topos (1/4)

A (commutative, unital) quantale is a complete lattice with a (commutative, unital) multiplication $\otimes$ such that: $a \otimes \bigvee_i b_i = \bigvee_i a \otimes b_i$.

Internal suplattices in a topos

An *internal suplattice* in a topos $\mathcal{E}$ is an object X equipped with a join operation $\bigvee : \mathcal{P}X \to X$.[1]

Internal quantale in a topos (1/2)

A (commutative, unital) *internal quantale* is an internal commutative monoid (X, μ, e) and internal suplattice $(X, \bigvee)$ structure, satisfying:

$$\begin{tikzcd}
\mathcal{P}X \times X \arrow[r, "t_{X,X}"] \arrow[d, "\bigvee \times 1_X"'] & \mathcal{P}(X \times X) \arrow[r, "\mathcal{P}\mu"] & \mathcal{P}X \arrow[d, "\bigvee"] \\
X \times X \arrow[rr, "\mu"'] & & X
\end{tikzcd}$$

Note $t_{X,X} : \mathcal{P}X \times X \to \mathcal{P}(X \times X)$ sends $(A, b) \mapsto \{(a, b) \mid a \in A\}$.

Quantales internal to a topos (2/4)

Internal quantale in a topos (1/2)

A (commutative, unital) *internal quantale* is an internal commutative monoid (X, μ, e) and internal suplattice $(X, \bigvee)$ structure, satisfying:

$$\begin{tikzcd} \mathcal{P}X \times X \arrow[r, "t_{X,X}"] \arrow[d, "\bigvee \times 1_X"'] & \mathcal{P}(X \times X) \arrow[r, "\mathcal{P}\mu"] & \mathcal{P}X \arrow[d, "\bigvee"] \\ X \times X \arrow[rr, "\mu"'] & & X \end{tikzcd}$$

Example: lifting monoid structure on X to quantale product on $\mathcal{P}X$

If $(X, +, 0)$, then $(\mathcal{P}X, \bigcup, \otimes_{\text{Mink}}, \{0\})$ is an internal quantale.

$$A \otimes_{\text{Mink}} B = \{a + b \mid a \in A, b \in B\}$$

Quantales internal to a topos (3/4)

Residuation morphism

Suppose we have a distinguished point: $I\colon 1 \to Q$.

There is a *residuation morphism* $(-)^*\colon \mathcal{Q} \to \mathcal{Q}$ given by $q \mapsto \bigvee\{x \mid x \otimes q \leq I\}$.[1]

These residuation morphisms have nice properties.

Closure operation

$(-)^{**}$ is a $\mathsf{Quant}_{\mathcal{E}}$ endomorphism and closure (idempotent, increasing) operation.

Quantales internal to a topos (4/4)

A sequence of forgetful functors:

$$\mathsf{Quant}_{\mathcal{E}} \longleftarrow \mathsf{PQ}_{\mathcal{E}} \xleftarrow[\text{wide}]{} \mathsf{PQ}_{\mathcal{E}}^{\mathsf{cont}} \xleftarrow[\text{full}]{} \mathsf{GQ}_{\mathcal{E}}$$

$$\overbrace{\begin{matrix} I\colon 1 \to Q \\ (-)^* = - \multimap I \end{matrix}} \qquad \overbrace{\begin{matrix} j = (-)^{**} \\ j_X \cdot f \sqsubseteq f \cdot j_Y \end{matrix}} \qquad \overbrace{\begin{matrix} j = \mathrm{id}_Q \\ x \parr y = (x^* \otimes y^*)^* \end{matrix}}$$

> Girard quantales are a reflective subcategory

$$\begin{tikzcd} \mathsf{PQ}_{\mathcal{E}}^{\mathsf{cont}} \arrow[r, bend left, "\rho"] \arrow[r, phantom, "\perp"] & \mathsf{GQ}_{\mathcal{E}} \arrow[l, bend left, "\iota"] \end{tikzcd}$$

The free sequent-set functor

$$F^{\vdash} : \mathcal{E} \to \mathsf{Quant}_{\mathcal{E}}$$

$$X \longmapsto (\mathbb{N}[X], +, 0) \longmapsto (\mathbb{N}[X]^2, +^2, 0^2) \longmapsto (\mathcal{P}(\mathbb{N}[X]^2), \otimes_{\mathrm{Mink}}, \{(0,0)\}, \bigcup)$$

Free Commutative Monoid — Doubling — Free Join Completion

E.g. in Set, it sends a set X to the set of assignments of X-sequents[1] as good or bad.

$F^{\vdash}$ a left adjoint

Internal implication frames (1/4): definition

$$\begin{tikzcd}
\mathsf{IF}_{\mathcal{E}}^{\mathsf{cont}} \arrow[r, "\pi_{\mathsf{PQ}}"] \arrow[d, "\pi_{\mathcal{E}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{PQ}_{\mathcal{E}}^{\mathsf{cont}} \arrow[d, "\text{bifibration}"] \\
\mathcal{E} \arrow[r, "F^{\vdash}"'] & \mathsf{Quant}_{\mathcal{E}}
\end{tikzcd}$$

An object of $\mathsf{IF}_{\mathcal{E}}^{\mathrm{cont}}$ is a pair (X, I)

- An object $X \in \mathrm{Ob}\,\mathcal{E}$
- A sequent sub-object $I \rightarrowtail \mathbb{N}[X]^2$ (i.e. $1 \to F^{\vdash} X$)

Internal implication frames (2/4): example frame in Set

Some assumptions about a frame (X, I) to make it finitely representable:

$X = \{a, b\}$ Multiplicity doesn't matter for whether or not a sequent is good or not.

$\mathcal{P}(X)^2$	0	a^-	b^-	a^-b^-
0	$\vdash$	$\vdash a$	$\vdash b$	$\vdash a, b$
a^+	$a \vdash$	$a \vdash a$	$a \vdash b$	$a \vdash a, b$
b^+	$b \vdash$	$b \vdash a$	$b \vdash b$	$b \vdash a, b$
a^+b^+	$a, b \vdash$	$a, b \vdash b$	$\vdash$	$a, b \vdash a, b$

Internal implication frames (3/4): free pointed quantale

$$\begin{tikzcd}
\mathsf{IF}^{\mathsf{cont}}_{\mathcal{E}} \arrow[r, "\pi_{\mathsf{PQ}}"] \arrow[d, "\pi_{\mathcal{E}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{PQ}^{\mathsf{cont}}_{\mathcal{E}} \arrow[d, "\text{bifibration}"] \\
\mathcal{E} \arrow[r, "F^{\vdash}"'] & \mathsf{Quant}_{\mathcal{E}}
\end{tikzcd}$$

The pullback of a left adjoint along a fibration is itself a left adjoint, so π_{PQ} is a left adjoint!

Internal implication frames (4/4): free Girard quantale

$$
\begin{tikzcd}
\mathsf{IF}^{\mathsf{cont}}_{\mathcal{E}} \arrow[r, "\pi_{\mathsf{PQ}}"] \arrow[d, "\pi_{\mathcal{E}}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{PQ}^{\mathsf{cont}}_{\mathcal{E}} \arrow[d, "\text{bifibration}"] \\
\mathcal{E} \arrow[r, "F^{\vdash}"'] & \mathsf{Quant}_{\mathcal{E}}
\end{tikzcd}
$$

An object of $\mathsf{IF}^{\mathrm{cont}}_{\mathcal{E}}$ is a pair (X, I)

- An object $X \in \mathrm{Ob}\,\mathcal{E}$
- A sequent sub-object $I \rightarrowtail \mathbb{N}[X]^2$ (i.e. $1 \to F^{\vdash} X$)

Compose free pointed quantale with GQ reflective subcategory adjunction:

$$
\begin{tikzcd}
\mathsf{IF}^{\mathsf{cont}}_{\mathcal{E}} \arrow[r, bend left, "\pi_{\mathsf{PQ}}"] \arrow[r, phantom, "\perp"] & \mathsf{PQ}^{\mathsf{cont}}_{\mathcal{E}} \arrow[l, bend left] \arrow[r, bend left, "\rho"] \arrow[r, phantom, "\perp"] & \mathsf{GQ}_{\mathcal{E}} \arrow[l, bend left, hook', "\iota"]
\end{tikzcd}
$$

Semantic consequence relation

The unit is $\eta_{\mathcal{X}} : \mathcal{X} \to \hat{\mathcal{X}}$ where $\hat{\mathcal{X}} := (\hat{X}, \hat{I})$ is the corresponding *semantic frame*.

$$\hat{X} = \{(A, B) \mid A, B \in F^{\vdash} X, A = A^{**}, B = B^{**}\}$$

Semantic values are pairs of sub-sequent objects (*premisory role* and *conclusory role*)

- Only the former is relevant to consequence when the value is a premise
- Only the latter is relevant to consequence when the value is a conclusion

Example semantic consequence between two arbitrary premises + conclusions

$$\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \vDash \langle c_+, c_- \rangle, \langle d_+, d_- \rangle \iff a_+ \otimes b_+ \otimes c_- \otimes d_- \subseteq I$$

Semantic consequence relation: conservativity

The unit is $\eta_{\mathcal{X}} : \mathcal{X} \to \hat{\mathcal{X}}$ where $\hat{\mathcal{X}} := (\hat{X}, \hat{I})$ is the corresponding *semantic frame*.

$$\hat{X} = \{(A, B) \mid A, B \in F^{\vdash} X, A = A^{**}, B = B^{**}\}$$

The unit is a $\mathcal{E}$ morphism, thought of as assigning semantic values.

Property of conservativity

$$\Gamma \vdash \Delta \iff [\![\Gamma]\!] \vDash [\![\Delta]\!]$$

Defining operations on the semantic space

For any semantic frame $(\hat{X}, \hat{I})$ we can intelligibly apply the following operations:

Formula	Interpretation	Formula	Interpretation
$[\![\neg A]\!]$	$\langle \mathtt{a}_-, \mathtt{a}_+ \rangle$	$[\![A \wedge B]\!]$	$\left\langle \begin{array}{c} \mathtt{a}_+ \otimes \mathtt{b}_+, \\ \mathtt{a}_- \wedge \mathtt{b}_- \wedge (\mathtt{a}_- \otimes \mathtt{b}_-) \end{array} \right\rangle$
$[\![A \otimes B]\!]$	$\langle \mathtt{a}_+ \otimes \mathtt{b}_+, \mathtt{a}_+ \parr \mathtt{b}_+ \rangle$	$[\![A \oplus B]\!]$	$\langle \mathtt{a}_+ \vee \mathtt{b}_+, \mathtt{a}_- \wedge \mathtt{b}_- \rangle$

We obtain other operations by DeMorgan duals:

- classical disjunction $[\![A \vee B]\!] = [\![\neg(\neg A \wedge \neg B)]\!]$
- multiplicative disjunction $[\![A \parr B]\!] = [\![\neg(\neg A \otimes \neg B)]\!]$
- additive conjunction $[\![A \& B]\!] = [\![\neg(\neg A \oplus \neg B)]\!]$

Special kinds of implication frames

The semantic space is often rich enough to interpret flavors of propositional logic.

Reflexive implication frames

- contain all sequents $x \vdash x$
- Semantic consequence is supralinear

Formula	Interpretation
$[\![A \otimes B]\!]$	$\langle \mathtt{a}_+ \otimes \mathtt{b}_+, \mathtt{a}_+ \invamp \mathtt{b}_+ \rangle$
$[\![A \oplus B]\!]$	$\langle \mathtt{a}_+ \vee \mathtt{b}_+, \mathtt{a}_+ \wedge \mathtt{b}_+ \rangle$

Indefeasibly-reflexive + contractive frames

- contain all sequents $\Gamma, x \vdash x, \Delta$
- $\Gamma, x \vdash \Delta \iff \Gamma, x, x \vdash \Delta$
- Semantic consequence is supraclassical

Formula	Interpretation
$[\![A \wedge B]\!]$	$\left\langle \begin{array}{c} \mathtt{a}_+ \otimes \mathtt{b}_+, \\ \mathtt{a}_- \wedge \mathtt{b}_- \wedge (\mathtt{a}_- \otimes \mathtt{b}_-) \end{array} \right\rangle$

Note: these supra-linear/classical frames need not satisfy monotonicity or even transitivity!

Takeaway: look at the structure of your domain you're trying to elucidate with semantic analysis. Depending on its features, some logics may be fit/unfit for purpose.

Extra: Idempotent Example (1/3)

Let $\mathcal{X} := (X, I)$ where $X = \{a, b\}$ and I is given by the following table.

E.g. $a \vdash a, b$ and $a \nvdash b$ for this frame.

I	0	a^-	b^-	a^-b^-
0	✓	✓	×	✓
a^+	×	✓	×	✓
b^+	×	×	✓	✓
a^+b^+	✓	✓	✓	✓

Here is an individual $(-)^*$ computation

$\{a^+\}*$	0	a^-	b^-	a^-b^-
0	×	✓	×	✓
a^+	×	✓	×	✓
b^+	✓	✓	✓	✓
a^+b^+	✓	✓	✓	✓

Here are all of the singletons $(\{-\})^*$:

$(\{-\})^*$	0	a^-	b^-	a^-b^-
0	I	X_b	$X_\pm$	$\top$
a^+	$X_\pm$	$\top$	$X_\pm$	$\top$
b^+	$X_\mp$	$X_\mp$	$\top$	$\top$
a^+b^+	$\top$	$\top$	$\top$	$\top$

$$X_\pm := \top \setminus \mathcal{P}[\{a^+, b^-\}] \qquad X_b := \top \setminus \{b^+, b^+a^-\}$$

$$X_\mp := \top \setminus \mathcal{P}[\{a^-, b^+\}] \qquad \top := \mathcal{P}[X + X]$$

Extra: Idempotent Example (2/3)

Now we can derive more inferential roles in $\mathfrak{G}$ by taking intersections of the singleton roles from the previous table, but this just yields one new role $X_\perp = \{a^+, b^+\}^\perp = X_\pm \cap X_\mp$.

Extra: Idempotent Example (3/3)

We have base cases:

$$[\![a]\!] = \langle \{a^+\}^{**}, \{a^-\}^{**} \rangle = \langle X_\mp, I \rangle \qquad [\![b]\!] = \langle \{b^+\}^{**}, \{b^-\}^{**} \rangle = \langle X_\pm, X_\mp \rangle$$

We can use the formula for semantic consequence to show that: $[\![a]\!], [\![b]\!] \vDash [\![a \wedge b]\!]$

$$\begin{aligned}
\pi_1([\![a]\!]) \otimes \pi_1([\![b]\!]) \otimes \pi_2([\![a \wedge b]\!]) &\subseteq I \\
\pi_1([\![a]\!]) \otimes \pi_1([\![b]\!]) \otimes \pi_2([\![a]\!]) \vee \pi_2([\![b]\!]) \vee (\pi_2([\![a]\!]) \otimes \pi_2([\![b]\!])) &\subseteq I \\
X_\mp \otimes X_\pm \otimes (I \vee X_\mp \vee (I \otimes X_\mp)) &\subseteq I \\
X_\mp \otimes X_\pm \otimes X_b &\subseteq I \\
X_\perp &\subseteq I
\end{aligned}$$

$\mathcal{X}$ is indefeasibly reflective+contractive, so $\vDash$ is *supraclassical* (but **not** monotonic).

Computer implementation

These calculations can be hairy but can be mechanized: ROLE.jl

```
"""
a = 'Zazzles the cat has four legs',
b = 'Zazzles the cat lost a leg'

|     |   | a | b | a,b |
|-----|---|---|---|-----|
|     | ✓ | ✓ | × |   ✓ |
|   a | × | ✓ | × |   ✓ |
|   b | × | × | ✓ |   ✓ |
| a,b | ✓ | ✓ | ✓ |   ✓ |
"""
C = ImpFrame([[]=>[:a], []=>[:a,:b], [:a,:b]=>[]], [:a,:b]; containment=true)
𝕒, 𝕓 = contents(C)
∅ = typeof(𝕒)[]               # empty list of contents
@test ∅ ⊩ (((𝕒 → 𝕓) → 𝕒) → 𝕒) # pierce's law
@test ∅ ⊮ ((𝕒 → 𝕓) → 𝕒)        # not pierce's law
```

Especially when considering contractive frames (finite set of candidate implications).

Technical Outline

~~1. Internal (pointed, Girard) quantales~~

(*Hidden: deriving reflector of Girard quantale reflective subcategory*)

~~2. The ‘free sequent-set’ functor~~

(*Hidden: unit and counit formulas*)

~~3. Implication frames and the logical completion~~

(*Hidden: why model radically-substructural reason relations?*)

~~4. Interpreting MALL and classical logic in a frame~~

(*Hidden: Supralinearity proof*)

(*Hidden: Nonmonotonic Multisuccedent Sequent calculus*)

5. Nominal sets to model subsentential structure

The category of nominal sets

Two ways of looking at **Nom** the category of nominal sets:

Nominal sets as actions

Let $\mathbb{A} = \{a, b, c, ...\}$ be a countably infinite set.

A *nominal set* is a finitely-supported $\mathrm{Perm}\ \mathbb{A}$ action.

- Each element $x \in X$ has a finite support, the variables it *depends* on.
- If the support is fixed by some permutation π, then π's action fixes x

Nominal sets as copresheaves

Let $\mathbb{I}$ be the category of finite sets and injective maps.

A *nominal set* is a pullback-preserving functor $\mathbb{I} \to$ **Set**.

$$\Delta \mapsto \{\text{subset of } X \text{ which is supported by } \Delta\}$$

What is an implication frame internal to Nom

It is a nominal set X equipped with a sub-nominal $I \rightarrowtail \mathbb{N}[X]^2$.

- "Claimables" are now allowed to depend on parameters: e.g. ψ, P_a, P_b, Q_{ab}
- Sequent components can share variables: e.g. $P_a \vdash Q_{ab}$ and $Q_{ba}, P_a \nvdash Q_{ab}$

I still picks out the subset of good sequents, but the choice must be *equivariant*:

- label names cannot matter: $P_a \vdash Q_{ab} \iff P_b \vdash Q_{ba} \iff P_a \vdash Q_{ac}$
- notation to help with this: $P_- \vdash Q_{-=}$

What is the free Girard quantale in Nom

Definition: finitely-supported subset of a nominal set

A subset of nominal set is supported by Γ if it is unchanged by permutations that fix Γ, applied pointwise to the elements of the subset.

- Any nominal subset supported by $\varnothing$ because it's equivariant.
- The subset $\{P_a, P_b\}$ is not equivariant but it is finitely supported by $\{a, b\}$.
- The infinite subset $\{P_a, P_c, P_d, \ldots\}$ is **not** finitely supported.

Let $\mathfrak{G}$ be the free internal Girard quantale for $(X, I) \in \mathsf{IF}_{\mathsf{Nom}}$.

- These are the $(-)^{**}$-closed f.s. subsets of $\mathbb{N}[X]^2$.

We need more than just Girard structure on $\mathfrak{G}$ in order to define $[\![\forall a\colon\ \Phi(\Gamma, a)]\!]$ on $\mathfrak{G}^2$ elements.

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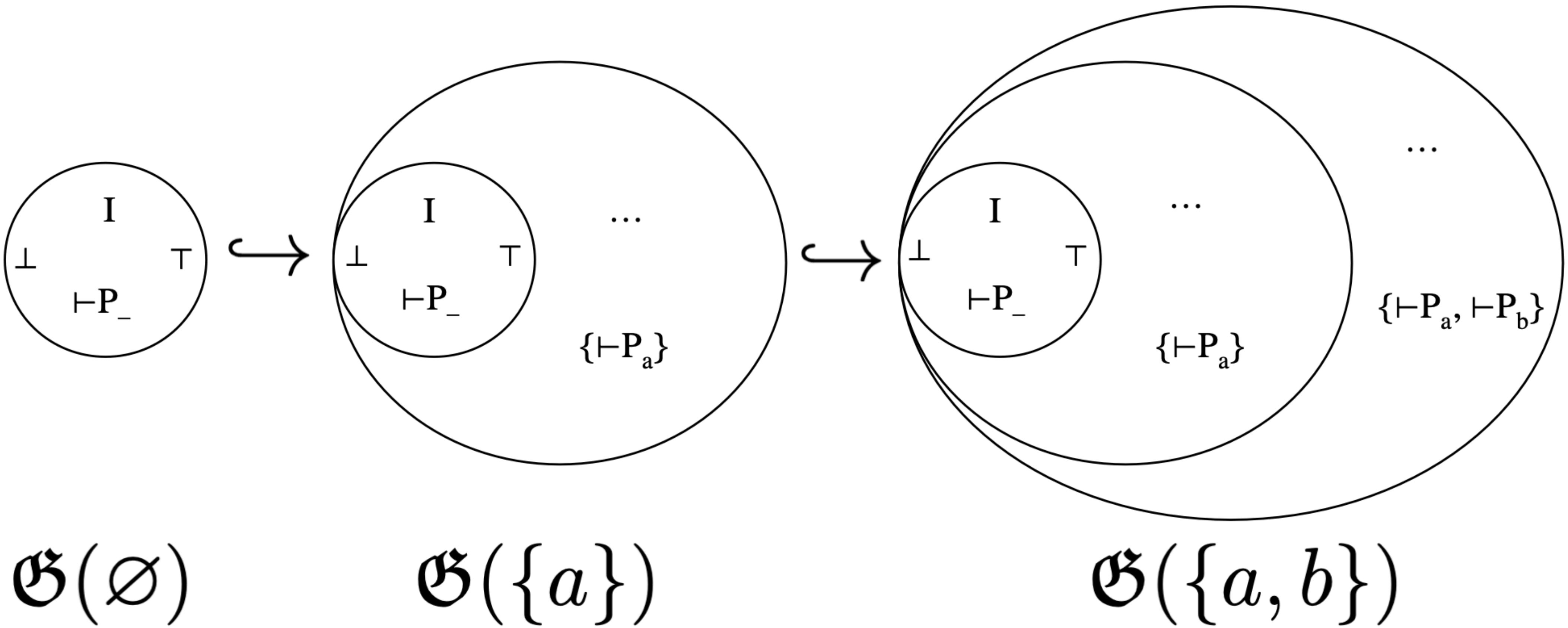
MALL Hyperdoctrine

If we view $\mathfrak{G}$ from the indexed perspective, we have a *functor* $\mathfrak{G} : \mathbb{I} \to \mathsf{GQ}$.

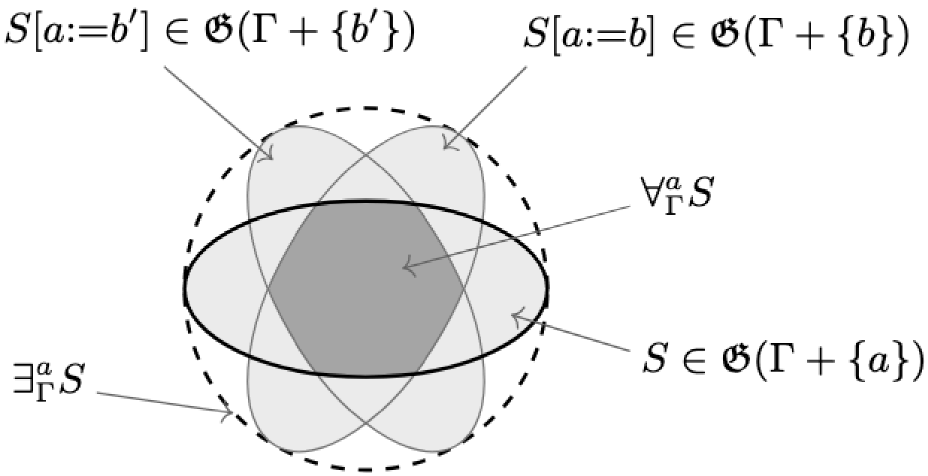


This is the data of a **MALL hyperdoctrine** if some other properties obtain:

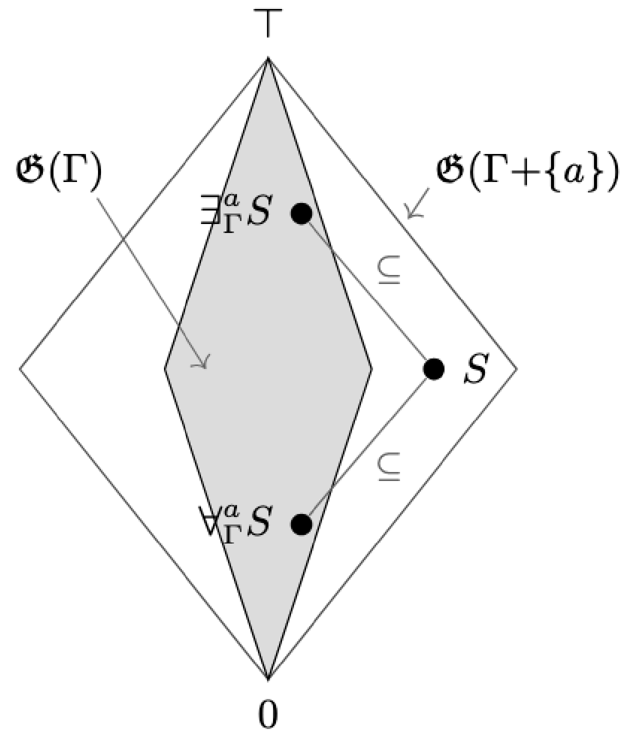
- Left/right adjoints for each $\Gamma \rightarrowtail \Delta$
- Frobenius reciprocity
- Beck-Chevalley

MALL Hyperdoctrine: adjoints

Let $S \in \mathfrak{G}(\Gamma + \{a\})$ and let $\iota : \Gamma \hookrightarrow \Gamma + \{a\}$.



the orbit $\{S[a{:=}b] \mid b \notin \Gamma\}$ of S:
$\forall$ = what is shared by all renamings
$\exists$ = the $\perp\perp$-closure of their union



S is between roles supported by Γ:
$\forall^a_\Gamma S$ = greatest such role below S
$\exists^a_\Gamma S$ = least such role above S

> **Formula for right adjoint to inclusion of Girard quantales:** $\forall^a_\Gamma(S) = \bigcap_{b \notin \Gamma} S[a := b]$

MALL Hyperdoctrine: Beck-Chevalley

Let $\iota : \Gamma \rightarrowtail \Delta$ and $S \in \mathfrak{G}(\Gamma + \{a\})$.

The Beck-Chevalley condition[1] is to the right:

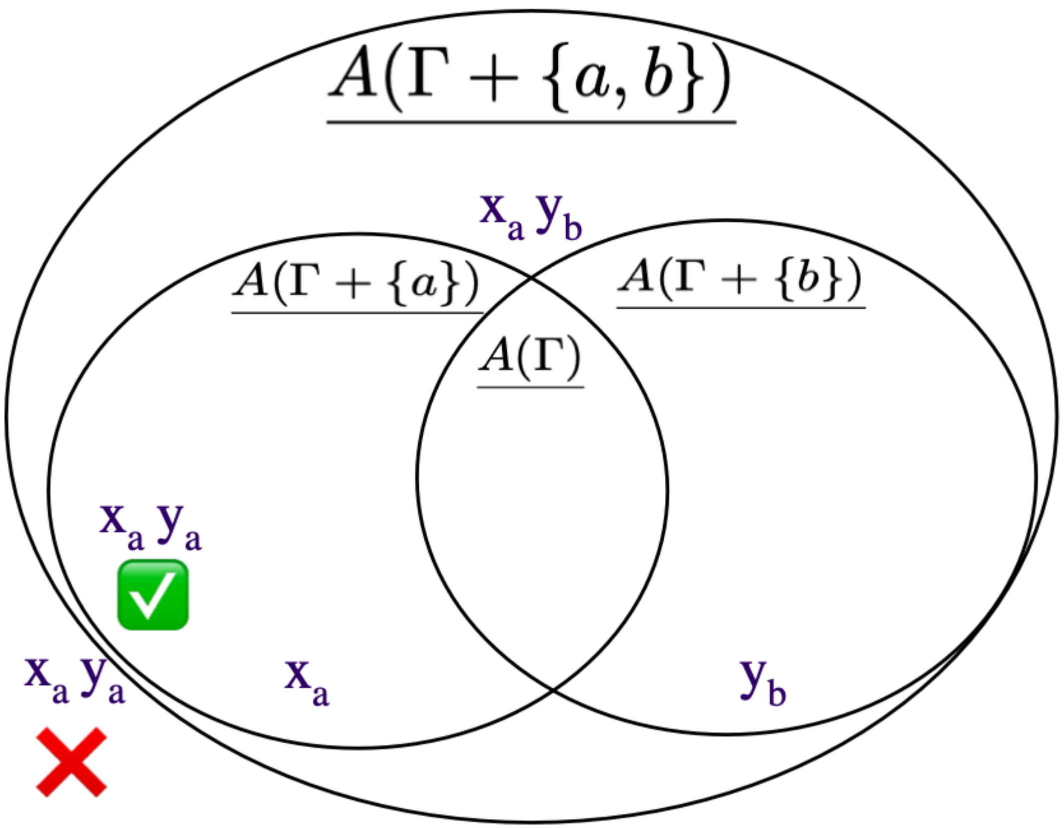
$$\begin{tikzcd} \mathfrak{G}(\Gamma + \{a\}) \arrow[r, "\mathfrak{G}(\iota + \{a\})"] \arrow[d, "\forall^a_\Gamma"'] & \mathfrak{G}(\Delta + \{a\}) \arrow[d, "\forall^a_\Delta"] \\ \mathfrak{G}(\Gamma) \arrow[r, "\mathfrak{G}(\iota)"'] & \mathfrak{G}(\Delta) \end{tikzcd}$$

This property holds if the subobject I in $(X, I \rightarrowtail \mathbb{N}[X]^2)$ is **substitution-equivariant**.

Substitution-equivariance (1/2)

Substitution equivariance of a monoid subobject in Nom

$A \rightarrowtail M$, is *substitution-equivariant* iff closed under identifying names in the following sense. Suppose $\{a, b\} \subseteq \Gamma$ with let $x_a \in M(\Gamma \setminus \{b\})$ and $y_b \in M(\Gamma \setminus \{a\})$ and $x_a y_b \in A(\Gamma)$. Substitution-equivariance means that $x_a y_a \in A(\Gamma \setminus \{b\})$, where $y_a := y_b[b{:=}a]$.



Substitution-equivariance (2/2)

Substitution equivariance of a monoid subobject in Nom

$A \rightarrowtail M$, is *substitution-equivariant* iff closed under identifying names in the following sense. Suppose $\{a, b\} \subseteq \Gamma$ with let $x_a \in M(\Gamma \setminus \{b\})$ and $y_b \in M(\Gamma \setminus \{a\})$ and $x_a y_b \in A(\Gamma)$. Substitution-equivariance means that $x_a y_a \in A(\Gamma \setminus \{b\})$, where $y_a := y_b[b{:=}a]$.

Substitution equivariance nonexample in implication frames

Consider Roberts Rules of Order, which states a motion must be seconded for the motion to be considered. We encode this norm in an implication frame.

$$\text{Motion}(a, m), \text{Second}(b, m) \vdash \text{Considered}(m)$$

This is an element of I. But we do *not* want the following in I, which is required by sub. equivariance:

$$\text{Motion}(a, m), \text{Second}(a, m) \vdash \text{Considered}(m)$$

Therefore we should allow the possibility of frames which are not substitutionally-equivariant.

Interpretations of quantifiers[1]

For the semantic clause, let $[\![\phi(\Gamma; a)]\!]$ have $\mathbf{p}$ as a premisory role and $\mathbf{c}$ as a conclusory role.

$$[\![\forall a\colon \phi(\Gamma; a)]\!] := \langle \forall^a_\Gamma(\mathbf{p}),\ \exists^a_\Gamma(\mathbf{c})\rangle \qquad \equiv \qquad [\![\mathop{\&}_{b \notin \Gamma} \phi(\Gamma; a)[a := b]]\!]$$

Now we can *logically* say what it means for a candidate implication to be in I:

$$a_1, \ldots, a_n \vdash_\Gamma b_1, \ldots, b_m \qquad \Longleftrightarrow \qquad \vDash \forall\Gamma\colon (a_1 \otimes \ldots \otimes a_n) \multimap (b_1 \parr \ldots \parr b_m)$$

The following rules depend on Beck Chevalley:

$$\boxed{\dfrac{\mathcal{A}, P \vdash_{\Gamma+\{x\}} \mathcal{B}}{\mathcal{A}, \forall x.P \vdash_\Gamma \mathcal{B}}\ \forall L} \qquad \boxed{\dfrac{\mathcal{A} \vdash_{\Gamma+\{x\}} P, \mathcal{B}}{\mathcal{A} \vdash_\Gamma \forall x.P, \mathcal{B}}\ \forall R}$$

Next steps

How to handle substitution more broadly?

- “Renaming sets” (subcategory of $[\mathbb{F}, \mathsf{Set}]$) (Gabbay and Hofmann 2008)

How to draw more connections to existing techniques in categorical logic.

Computational implementation: substructural knowledge bases

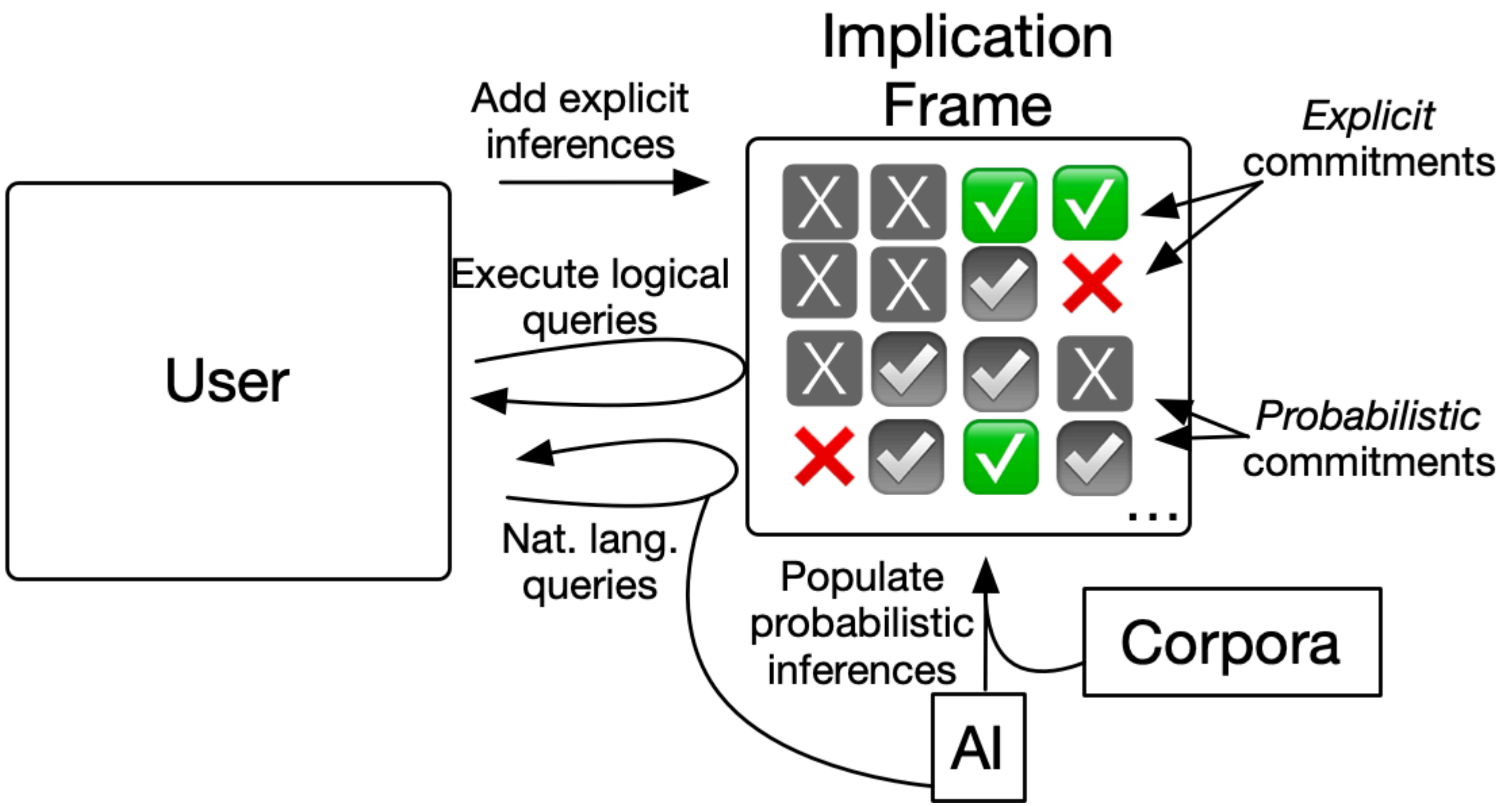
- Work with finite presentations of implication frames
- Use the implicit logical structure to build / query them
- Use the category of them to collaboratively / compositionally manipulate them

A vision for neurosymbolic AI

We want AI systems to have *some* kind of intelligible, interrogatable model.

At the same time, the lack of imposing structure (radical flexibility of present AI architectures) is practically useful.

Compromise: imp. frames have the right balance of unstructuredness and (latent) structure.



Implicit commitments
- use logical vocabulary
- transparently computed from the frame data

Conclusions: logical expressivism

We can model a **norm** where some claimables can be asserted and denied, and some combinations of assertions and denials are in-bounds or not.

- This is the data of a radically-substructural consequence relation.
- This is a model of the thing that we care about when we do formal modeling, including the assignment of informal concepts to formal constructs:
  - when we design formal calculi, a criterion of adequacy is that they reproduce inferences we already take ourselves to be entitled to make!

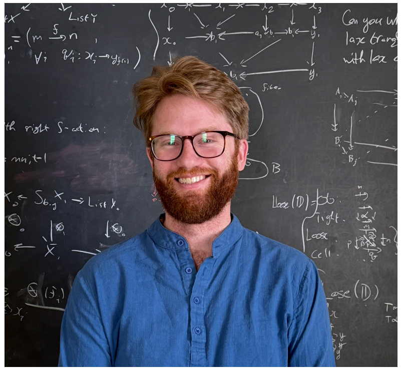
Even if we assume *nothing* else about the domain, we can *introduce* new claimables to be asserted or denied, built out of connectives of the old ones.

- This does not change the goodness of inference between the original claimables (*conservative* extension).
- When the norm meets some basic criteria, these connectives recover the inferential properties we expect from well known logics (MALL, classical logic).

By generalizing predicate logic, we can compactly represent an infinitude of judgments into a single sequent.

Thank you for listening

And even more thanks to:



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And the Research on Logical Expressivism (ROLE) group:

- Robert Brandom, Ulf Hlobil, Ryan Simonelli, Rea Golan, Shuhei Shimamura, and others.

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