Joint w/ John Francis

Topos Institute
Nov 2026

Valent Higher Groupoids

Ask Questions

Thm ("Full beta" Non-Abelian Poincaré Duality)

Let $\mathcal{G}$ be a valent ∞-groupoid.
Let M be a compact stratified space.
There is a canonical equivalence between spaces:

$$\int_M^{\mathcal{B}} \mathcal{G} \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Fil}(S)}\left(\mathrm{sk}(M), \mathcal{B}\mathcal{G}\right)$$

- $\int_M^{\mathcal{B}} \mathcal{G}$: Factorization Homology of $\mathcal{G}$ over M — a colimit ∞{state sum}
- $\mathrm{sk}(M)$: skeletal filtration
- $\mathrm{Fil}(S)$: Filtered spaces $\mathrm{Fun}(\mathbb{N}, S)$ — a limit ∞{sigma-model}

Cor For M a closed n-manifold,

$$\int_M \mathcal{G} \xrightarrow{\cong} \mathrm{Map}\left(M, |\mathcal{G}_{\le n}|\right)$$

$|\mathcal{G}_{\le n}|$: classifying space of the underlying valent n-groupoid of $\mathcal{G}$.

Eg For $\mathcal{G} = \mathcal{B}^n A$ for A an abelian group and for M closed framed n-manifold,

$$Sp^\infty(M;A) = \int_M \mathcal{B}^n A \xrightarrow{\cong} \mathrm{Map}(M, \mathcal{B}^n A)$$
$$\downarrow \pi_q \qquad\qquad \downarrow \pi_q$$
$$H_q(M;A) \underset{\text{Poincaré duality}}{\cong} H^{n-q}(M;A).$$

Eg For $\mathcal{G} = \mathcal{B}^n A$ for A an E_n-group, and for M closed framed n-manifold,

$$\int_M^{\alpha} A \simeq \int_M^{\mathcal{B}} \mathcal{B}^n A$$

$\rightsquigarrow$ Nonabelian Poincaré duality [Salvatore, Lurie, AFrancis]

Proof Outline

Today:

1) Prove **Thm** (Non-Linear Koszul Duality) (Full beta) The functor

$$\mathcal{B} : v\mathrm{Gpd}_\infty \longrightarrow \mathrm{Fil}(S) := \mathrm{Fun}(\mathbb{N}, S)$$
$$\mathcal{G} \longmapsto (|\mathcal{G}_{\le 0}| \to |\mathcal{G}_{\le 1}| \to \cdots)$$

is fully faithful w/ image those $B_0 \to B_1 \to \cdots$ s.t. $B_n \to B_{n+1}$ is n-connected $\forall n \in \mathbb{N}$.

2) $\int_M \mathcal{G} \equiv \mathrm{Colim}_{D \in \mathcal{D}(M)} \mathrm{Hom}_{\mathrm{Fil}(S)}(\mathrm{sk}(D), \mathcal{B}\mathcal{G})$
$\mathcal{D}(M)$: category of disk-refinements of M

3) For $\forall\ \tilde{M} \to M$ refinement, $\mathrm{sk}(\tilde{M}) \xrightarrow{\forall} \mathcal{B}\mathcal{G}$, $\mathrm{sk}(M) \xrightarrow{\exists}$ (via connectivity)

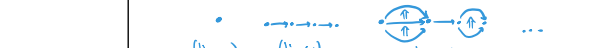
4) Prove $|\mathcal{D}(M)| = *$

Valent Higher Categories

Recall the full subcategory

$\Theta \subset \omega\mathrm{Cat}$ ← (strict) omega-categories

consisting of those that are free on a globular pasting diagram:



Restricted yoneda $\rightsquigarrow \omega\mathrm{Cat} \xrightarrow{N} \mathrm{PShv}(\Theta)$ (S = Space-valued)

$$\mathcal{C} \longmapsto \left(\Theta^{op} \ni T \mapsto \mathrm{Hom}_{\omega\mathrm{Cat}}(T, \mathcal{C})\right)$$

Fact N is fully faithful w/ image those $\mathcal{F} \in \mathrm{PShv}(\Theta)$ s.t.

Segal $\forall$ $T_0 \to T_1$, $T_0 \to T_2$ $\mapsto$ $\mathcal{F}(T) \to \mathcal{F}(T_1)$, $\mathcal{F}(T) \to \mathcal{F}(T_2)$, $\mathcal{F}(T_1) \to \mathcal{F}(T_{01})$, $\mathcal{F}(T_2) \to \mathcal{F}(T_{02})$
globular gluing in Θ

0-Type $\forall\ T \in \Theta$, $\mathcal{F}(T) \in S_{\le 0}$ is a 0-type

Image of $\omega\mathrm{Gpd} \hookrightarrow \omega\mathrm{Cat} \xrightarrow{N} \mathrm{PShv}(\Theta)$

Invertible $\mathcal{F}(C_n[2]) \to \mathcal{F}(C_n[0 \to 2])$, $\mathcal{F}(C_n[2]) \to \mathcal{F}(C_n[0 \to 1])$, $\mathcal{F}(C_n[0 \to 2]) \to \mathcal{F}(C_n[0 \to 1])$, $\mathcal{F}(C_n[0 \to 1]) \to \mathcal{F}(C_n[0])$
∞{every cell is invertible}

Def the $(\infty,1)$-category of valent (∞,∞)-categories is

$$v\mathrm{Cat}_\infty := \mathrm{PShv}^{Seg}(\Theta) \subset \mathrm{PShv}(\Theta)$$

and of valent ∞-groupoids is

$$v\mathrm{Gpd}_\infty := \mathrm{PShv}^{Seg, inv}(\Theta) \subset v\mathrm{Cat}_\infty.$$

Obs For each $n \in \mathbb{N}$, underlying valent n-cat/n-gpd:

$$(-)_{\le n} : v\mathrm{Cat}_\infty \rightleftarrows v\mathrm{Cat}_n \quad \text{and} \quad (-)_{\le n} : v\mathrm{Gpd}_\infty \rightleftarrows v\mathrm{Gpd}_n$$

Recall For each $n \in \mathbb{N}$, (univalent) n-categories:

$$\mathrm{Cat}_n \underset{(-)^{univ}}{\overset{\iota}{\rightleftarrows}} v\mathrm{Cat}_n \qquad \mathrm{Gpd} \simeq S \underset{(-)^{univ}}{\overset{\iota}{\rightleftarrows}} v\mathrm{Gpd}_n$$

Q How do valent higher groupoids arise?

A1 (strict) ω-groupoids:

$\omega\mathrm{Gpd} \hookrightarrow v\mathrm{Gpd}_\infty$ (discrete examples)

A1 (univalent) Groupoids ≡ Spaces:

$\mathrm{Gpd} \equiv S \hookrightarrow v\mathrm{Gpd}_\infty$, $B \mapsto \mathrm{Const}_B$ $(B \otimes B \otimes \cdots)$ obj Mor zero

A1 E_n-groups.

$\mathrm{Alg}^{gp}_{E_n}(S) \xrightarrow{\mathcal{B}^n} v\mathrm{Gpd}_\infty$, $A \mapsto \mathcal{B}^n A$ $(* \otimes \cdots \otimes * \otimes A \otimes A \otimes \cdots)$ $n-1$

(Alg) Non-Linear Koszul Duality: $\Omega^{[M_n]} \mathcal{B}^n : (S^{M_n})_{\ge n} \to \mathcal{B}^n$

$(n-1)$-connected

Flagged Spaces

Def the category of flagged ∞-categories is the full subcat

$$f\mathrm{Cat}_\infty \subseteq \mathrm{Fun}(\mathbb{N}, \mathrm{Cat}_\infty)$$

consisting of those $\mathcal{B}_0 \to \mathcal{B}_1 \to \cdots$ s.t.
- $\forall\ n \in \mathbb{N}$, $\mathcal{B}_n$ is an n-category
- $\forall\ n \in \mathbb{N}$, $\mathcal{B}_n \to \mathcal{B}_{n+1}$ is n-connected

$\forall\ 0 \le k \le n$: $2C_k \hookrightarrow C_k$

Schemes ff ↪ images ↪ $\mathrm{Sch}^{\mathbb{E}}_{\mathbb{E}}(\mathrm{CAlg})$ ↪ $\mathrm{PSh}(\mathrm{CAlg}^{op})$

Thm (Flagged Higher Categories)

$\mathrm{PShv}^{Seg}(\Theta) = v\mathrm{Cat}_\infty \xrightarrow{\cong} f\mathrm{Cat}_\infty$

$$\mathcal{C} \longmapsto \left((\mathcal{C}_{\le 0})^{\wedge}_{univ} \to (\mathcal{C}_{\le 1})^{\wedge}_{univ} \to \cdots\right)$$
$$\left(\Theta^{op} \ni T \mapsto \mathrm{Hom}_{f\mathrm{Cat}_\infty}(T_{\le \bullet}, \mathcal{B}_\bullet) \in S\right) \longleftarrow (\mathcal{B}_0 \to \mathcal{B}_1 \to \cdots)$$

Def the category of flagged spaces is the full subcat

$$fS \subset \mathrm{Fil}(S)$$

consisting of those $B_0 \to B_1 \to \cdots$ s.t. $\forall\ n \in \mathbb{N}$, $B_n \to B_{n+1}$ is n-connected.

Obs $fS \xrightarrow{ff} f\mathrm{Cat}_\infty$, $(B_0 \to B_1 \to \cdots) \mapsto (B_0 \to B_1 \to \cdots)$

Thm (B) NL KD

$$\begin{array}{ccc} v\mathrm{Gpd}_\infty & \underset{\Omega}{\overset{\mathcal{B}}{\rightleftarrows}} & fS \\ \downarrow & & \downarrow \\ v\mathrm{Cat}_\infty & \xrightarrow{\cong} & f\mathrm{Cat}_\infty \end{array}$$

where $\mathcal{B}\mathcal{G} = (|\mathcal{G}_{\le 0}| \to |\mathcal{G}_{\le 1}| \to \cdots)$

and $\Omega\mathcal{B}(T) = \mathrm{Hom}_{f\mathrm{Cat}_\infty}(\mathcal{G}_{\le \bullet} \to T_{\le \bullet} \to \cdots), (\mathcal{B}_0 \to \mathcal{B}_1 \to \cdots))$

$= \mathrm{Hom}_{\mathrm{Fil}(S)}((\Pi_{\le 0} \to \Pi_{\le 1} \to \cdots), (B_0 \to B_1 \to \cdots))$

$= \left\{ \begin{array}{ccccccc} \mathcal{B}^0 & \to & \mathcal{B}^1 & \to & \cdots & \to & \mathcal{B}^n \\ \downarrow & & \downarrow & & & & \downarrow \\ B_0 & \to & B_1 & \to & \cdots & \to & B_n \end{array} \right\}$

$T = C_n$

Eg For $B = (\underbrace{x \to \cdots \to x}_{n-1} \to X \rightrightarrows X \rightrightarrows \cdots)$,

$$\Omega\mathcal{B} = \mathcal{B}^n(\Omega^n X)$$

Rmk the valent ∞-groupoid $\Omega\mathcal{B}$ classifies deformations of $B = (B_0 \to B_1 \to \cdots)$.

Rmk $\Omega(X \overset{f}{\to} Y \rightrightarrows Y \rightrightarrows \cdots) =$ Čech Nerve of $X \overset{f}{\to} Y$

$= \check{N}_{p+1} f = \underbrace{X \times_Y \cdots \times_Y X}_{p+1}$

Generally, $\Omega\mathcal{B}$ is a sort of iterated Čech nerve.

Problem Does this NLKD hold in an $(\infty,1)$-topos besides $S \equiv$ Spaces?

Strict ω-groupoids

Revisiting work of Brown-Higgins and Ara

Def the category of typed flagged spaces is

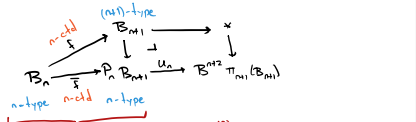
$$f\mathcal{S}^{typed} \subset f\mathcal{S}$$

those $B_0 \to B_1 \to \cdots$ s.t. $-\ \forall n \in \mathbb{N}$, B_n is a n-type
$-\ \forall n \in \mathbb{N}$, $B_n \to B_{n+1}$ is n-ctd.

Obs

$$\begin{array}{ccc} \omega\mathrm{Gpd} \equiv v\mathrm{Gpd}_\infty^{strict} & \underset{\Omega}{\overset{\mathcal{B}}{\rightleftarrows}} & f\mathcal{S}^{typed} \\ \downarrow & & \downarrow \\ v\mathrm{Gpd}_\infty & \underset{\Omega}{\overset{\mathcal{B}}{\rightleftarrows}} & f\mathcal{S} \end{array}$$

Rmk Let $\mathcal{B}$ be a typed flagged space.



1) $\pi_k(B_n) \xrightarrow{\pi_k(i)} \pi_k(P_n B_{n+1})$ iso for $k \le n$ (zero for $k > n$), surj for $k = n$

2) f witnesses vanishing of $\bar{f}^* [u_n] \in H^{n+2}(B_n; \pi_{n+1} B_{n+1})$

Prop $\forall$ abel gp C

$$H^{n+2}(P_n B_{n+1}; C) \xrightarrow{\bar{f}^*} H^{n+2}(B_n; C)$$

is injective.

Cor For $\mathcal{B}$ a typed flagged space w/ $B_0 = B_1 = *$,

$$B_n \cong \prod_{k \le n} \mathcal{B}^k \pi_k(B_k) \quad \forall n \in \mathbb{N}.$$

Cor For $\mathcal{G} \in \omega\mathrm{Gpd}$ w/ $\mathcal{G}_{\le 1} = *$,

$$\boxed{|\mathcal{G}| \cong \prod_{n \in \mathbb{N}} \mathcal{B}^n \pi_n(|\mathcal{G}|)}.$$

[In particular, all Whitehead products of $|\mathcal{G}|$ vanish.]

Reporting the E^1-page of the π_*-Spec Seq of the filtration $\mathcal{B} = (B_0 \to B_1 \to \cdots) \rightsquigarrow$

$$E^1 : f\mathcal{S}^{typed,\ t\text{-}ctd} \longrightarrow Ch_{\ge 2}(\mathrm{Ab})$$
$$\mathcal{B} \longmapsto \left(0 \leftarrow \cdots \leftarrow \pi_n(B_n) \xleftarrow{\pi_n(B_n \to B_{n+1})} \pi_n(B_{n+1}) \leftarrow \cdots\right)$$
$\pi_n B_n$, $\pi_n B_{n+1}$, $\pi_n B_{n+2}$

Reporting products of Eilenberg-MacLane spaces of homology

$\rightsquigarrow Ch_{\ge 2}(\mathrm{Ab}) \xrightarrow{\mathcal{B}} (S^{M_\bullet})_{\ge 2}$

$$E \longmapsto \prod_{n \in \mathbb{N}} \mathcal{B}^n H_n(E)$$

Incorporating (brutal) truncations

$\rightsquigarrow Ch_{\ge 2}(\mathrm{Ab}) \xrightarrow{\mathcal{B}^E} f\mathcal{S}^{typed,\ t\text{-}ctd}$

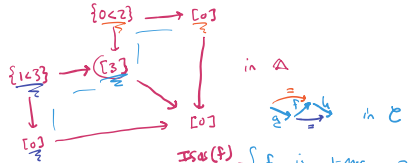
$$E \longmapsto \left(\cdots \to B(E_{\le n}) \to B(E_{\le n-1}) \to \cdots\right)$$

Thm

$$f\mathcal{S}^{typed,\ t\text{-}ctd} \underset{\mathcal{B}^E}{\overset{E^1}{\rightleftarrows}} Ch_{\ge 2}(\mathrm{Ab})$$

Cor $f\mathcal{S}^{typed,\ t\text{-}ctd} \xrightarrow[\text{Fully Faithful}]{\Omega^\infty} v\mathrm{Gpd}_\infty$!

$n=1$



$\mathrm{Isos}(\mathcal{F}) = \{ f \text{ is 1-mor} \mid \text{w/ L/R inverse} \}$

$\mathcal{F} \in \mathrm{PShv}(\Delta) \rightsquigarrow \mathcal{F}(\mathrm{Above})$

12

$\mathrm{Obj}_2(\mathrm{Morita}) = \mathrm{Alg}_S + \text{Bdds}$ ↓ Not an iso

$\mathrm{Iso}(\mathrm{Morita}) = \mathrm{Alg}_S + \text{Morita Eqvrs}$

$\mathrm{Obj}_2(\mathcal{C}) = \mathcal{C}[\omega]$

$\mathrm{Obj}_2(\mathcal{C}) \xrightarrow{\cong} \mathrm{Iso}(\mathcal{C})$

Univ Axiom