What are (∞,∞)-categories?

jt with Martina Rovelli & Tashi Walde

Main Theorem (Theorem 4.25). *There is a reflective localization*

$$L\colon \mathbf{Cat}^{R}_{(\infty,\infty)} \rightleftarrows \mathbf{Cat}^{L}_{(\infty,\infty)} : R$$

that inverts precisely the weakly ∞-surjective maps.

Analogy from algebra

$\mathbb{Z} \hookrightarrow \mathbb{Q}$

→ "invert non-zero integers"

→ "invert all primes"

What are $\mathbf{Cat}^{R}_{(\infty,\infty)}$ and $\mathbf{Cat}^{L}_{(\infty,\infty)}$?

Before: What is an (∞,n)-category?

$n=1$

(1)-categories

objects

(1-)morphisms

composition

$\to \mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor} \to \mathrm{Mor}$

$(\infty,1)$-categories

objects

1-morphisms

invertible { 2-morphisms

3-morphisms

⋮

- associative
- unital

composition
→ "implicit function"

- assoc up to higher cells
- unitality up to higher cells

$n = 2$

2_n-categories $\leq$ bicategory $\hookrightarrow$ $(\infty,2)$-categories

obj
1-morphisms
2-morphisms
composition

$1\text{Mor} \times_{\dots} 1\text{Mor} \to 1\text{Mor}$

$2\text{Mor} \times_{\dots} 2\text{Mor} \to 2\text{Mor}$

- assoc
- unital
- interchange laws

obj
1-morphisms
2-morphisms
3-
4-
$\leq n$-morphisms

inv { composition
→ "implicit function"

- assoc up to higher cells
- unitality up to higher cells
- interchange up to higher cells

What happens if $n \to \infty$?

→ Depends on functors

$$\operatorname{colim}_{\mathbb{N}^{\mathrm{op}}} (\infty,n)\mathrm{Cat} \;\overset{??}{\longrightarrow}\; (\infty,n+1)\mathrm{Cat}$$

$$\operatorname{colim}_{(\infty,0)\mathrm{Cat}} \left(\cdots \to (\infty,n)\mathrm{Cat} \underset{\text{incl}}{\overset{1}{\rightleftarrows}} (\infty,n+1)\mathrm{Cat} \to \cdots \right)$$

tr_n = "inverts $(n+1)$-morphisms"

core_n = "discards non-inv $(n+1)$-morphisms"

$$\operatorname{colim}^{\mathrm{Pr}^L} \left((\infty,0)\mathrm{Cat} \xrightarrow{\text{incl}} (\infty,1)\mathrm{Cat} \xrightarrow{\text{incl}} \cdots \right)$$

$$= \underline{\lim}^{(\infty,1)\mathrm{Cat}} \left((\infty,0)\mathrm{Cat} \xleftarrow[\text{core}]{} (\infty,1)\mathrm{Cat} \xleftarrow[\text{core}]{} \cdots \right)$$

$$=: \mathrm{Cat}^{R}_{(\infty,\infty)} \qquad = \lim^{\mathrm{Pr}^R}$$

(∞,∞)-category of bordisms

spans

Morita category

$$\mathrm{Cat}^{L}_{(\infty,\infty)} := \lim^{(\infty,1)\mathrm{Cat}} \left((\infty,0)\mathrm{Cat} \overset{\mathrm{tr}}{\leftarrow} (\infty,1)\mathrm{Cat} \overset{\mathrm{tr}}{\leftarrow} \cdots \right)$$

$$= \operatorname{colim}^{\mathrm{Pr}^R}\left((\infty,0)\mathrm{Cat} \xrightarrow[\text{incl}]{} (\infty,1)\mathrm{Cat} \xrightarrow[\text{incl}]{} \cdots\right)$$

$$\begin{array}{ccc} \mathrm{Cat}^L_{(\infty,n)} & \dashrightarrow & \mathrm{Cat}^R_{(\infty,\infty)} \\ \vdots & & \vdots \\ \text{colim} & & \lim \end{array}$$

$$\begin{array}{ccc} (\infty,k)\mathrm{Cat} & \longrightarrow & (\infty,\ell)\mathrm{Cat} \\ & & \begin{cases} \text{incl} & k \le \ell \\ \text{core} & k \ge \ell \end{cases} \end{array} \quad \Big\} \; \mathrm{Pr}^R$$

Check: This assembles in Pr^R, into

$$\begin{array}{ccc} \mathrm{Cat}^L_{(\infty,\infty)} & \xrightarrow{R} & \mathrm{Cat}^R_{(\infty,\infty)} \\ \uparrow & \mathrm{Pr}^L \not\simeq \int \mathrm{Loc} & \uparrow \\ \mathrm{Cat}^R_{(\infty,n)}\left[\Sigma^?\,\mathrm{Coind}\mathrm{Eq}^{-1}\right] & & \mathrm{Cat}^R_{(\infty,n)}\left[\Sigma^?\,\mathrm{Coind}\mathrm{Eq}^{-1}\right] \\ \neq \quad \neq \int \mathrm{Loc} & & \neq \quad \neq \int \mathrm{Loc} \\ \mathrm{Cat}^R_{(\infty,0)}\left[\Sigma^?\,\mathrm{Coind}\mathrm{Eq}^{-1}, \Sigma^?\,\mathrm{HL\text{-}example}^{-1}\right] & & \end{array}$$