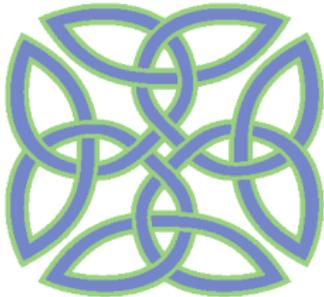


The many faces of magnitude

Tom Leinster



School of Mathematics
University of Edinburgh



Boyd Orr Centre
for Population and Ecosystem Health
University of Glasgow

Plan

1. Magnitude: the big picture
2. The magnitude of a metric space
- $2\frac{1}{2}$. The magnitude of a graph
3. Magnitude homology
4. (Bio)diversity

1. Magnitude: the big picture

The idea

For many types of mathematical object, there is a canonical notion of size.

- Sets have cardinality. It satisfies

$$|S \cup T| = |S| + |T| - |S \cap T|$$

$$|S \times T| = |S| \times |T|.$$

- Subsets of \mathbb{R}^n have volume. It satisfies

$$\text{vol}(S \cup T) = \text{vol}(S) + \text{vol}(T) - \text{vol}(S \cap T)$$

$$\text{vol}(S \times T) = \text{vol}(S) \times \text{vol}(T).$$

- Topological spaces have Euler characteristic. It satisfies

$$\chi(S \cup T) = \chi(S) + \chi(T) - \chi(S \cap T) \quad (\text{under hypotheses})$$

$$\chi(S \times T) = \chi(S) \times \chi(T).$$

Stephen Schanuel:

Euler characteristic is the topological analogue of cardinality.



The idea

For many types of mathematical object, there is a canonical notion of size.

- Sets have cardinality. It satisfies

$$|S \cup T| = |S| + |T| - |S \cap T|$$

$$|S \times T| = |S| \times |T|.$$

- Subsets of \mathbb{R}^n have volume. It satisfies

$$\text{vol}(S \cup T) = \text{vol}(S) + \text{vol}(T) - \text{vol}(S \cap T)$$

$$\text{vol}(S \times T) = \text{vol}(S) \times \text{vol}(T).$$

- Topological spaces have Euler characteristic. It satisfies

$$\chi(S \cup T) = \chi(S) + \chi(T) - \chi(S \cap T) \quad (\text{under hypotheses})$$

$$\chi(S \times T) = \chi(S) \times \chi(T).$$

Challenge Find a general definition of 'size', including these and other examples.

One answer The **magnitude of an enriched category**.

The magnitude of a matrix

Let Z be a matrix.

If Z is invertible, the **magnitude** of Z is

$$|Z| = \sum_{i,j} (Z^{-1})_{ij}$$

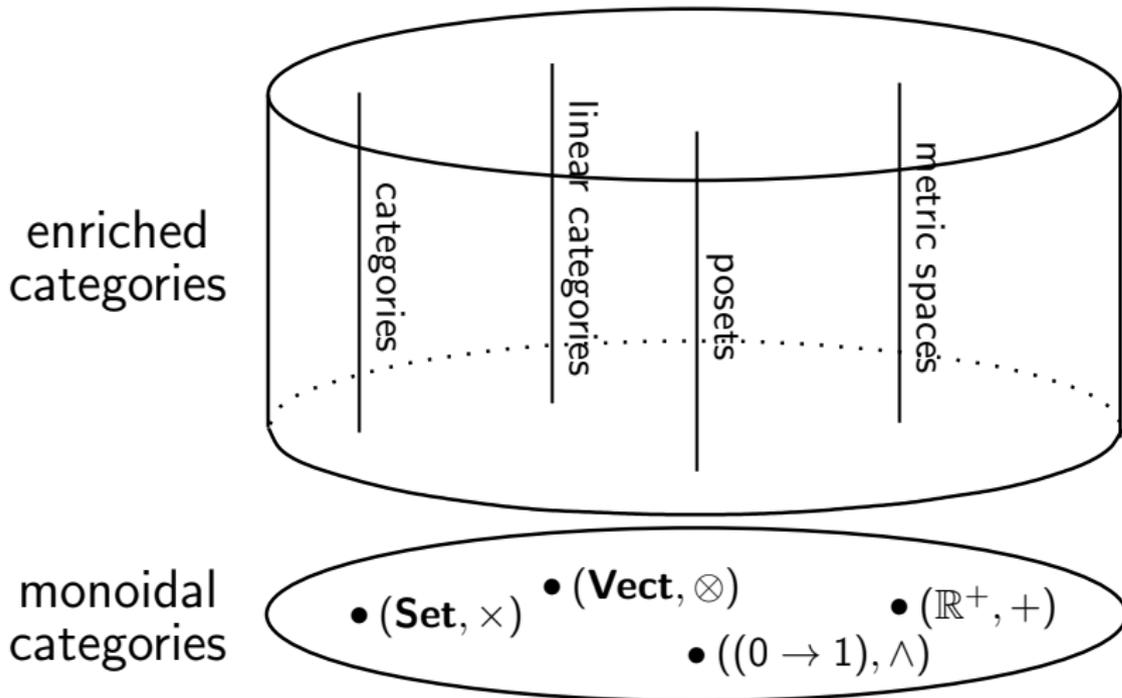
—the sum of all the entries of Z^{-1} .

(The definition can be extended to many non-invertible matrices. . . but we won't need this refinement today.)

Enriched categories

A **monoidal category** is a category \mathbf{V} equipped with some kind of product.

A **category enriched in \mathbf{V}** is like an ordinary category, with a set/class of objects, but the 'hom-sets' $\text{Hom}(A, B)$ are now objects of \mathbf{V} .



The magnitude of an enriched category

Let \mathbf{V} be a monoidal category.

Suppose we have a notion of the ‘size’ of each object of \mathbf{V} :
a multiplicative function $|\cdot|$ from $\text{ob } \mathbf{V}$ to some field k .

E.g. $\mathbf{V} = \mathbf{FinSet}$, $k = \mathbb{Q}$, $|\cdot| = \text{cardinality}$;

$\mathbf{V} = \mathbf{FDVect}$, $k = \mathbb{Q}$, $|\cdot| = \text{dimension}$.

Then we get a notion of the ‘size’ of a category \mathbf{A} enriched in \mathbf{V} :

- write $Z_{\mathbf{A}}$ for the matrix $(|\text{Hom}(A, B)|)_{A, B \in \text{ob } \mathbf{A}}$ over k
- define the **magnitude** of the enriched category \mathbf{A} to be

$$|\mathbf{A}| = |Z_{\mathbf{A}}| \in k$$

—i.e. the magnitude of the matrix $Z_{\mathbf{A}}$.

(Here assume \mathbf{A} has only finitely many objects and $Z_{\mathbf{A}}$ is invertible.)

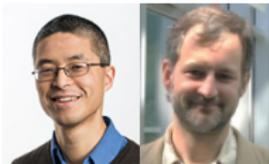
Examples *not* involving metric spaces

Ordinary finite categories (i.e. $\mathbf{V} = \mathbf{FinSet}$):

- For a finite category \mathbf{A} satisfying mild conditions, $|\mathbf{A}|$ is $\chi(B\mathbf{A}) \in \mathbb{Z}$, the Euler characteristic of the classifying space of \mathbf{A} .
- For a finite group G seen as a one-object category, $|G| = 1/\text{order}(G)$.
- For a finitely triangulated manifold X , its poset \mathbf{A} of simplices has magnitude $|\mathbf{A}| = \chi(X)$.

Linear categories (i.e. $\mathbf{V} = \mathbf{Vect}$):

- For a suitably finite associative algebra E , let $\mathbf{IP}(E)$ denote the linear category of indecomposable projective E -modules.



Theorem (with Joe Chuang and Alastair King)

The magnitude of $\mathbf{IP}(E)$ is a certain Euler form associated with E .

Metric spaces as enriched categories

There's at least an *analogy* between categories and metric spaces:

A category has:

objects a, b, \dots

sets $\text{Hom}(a, b)$

composition operation

$$\text{Hom}(a, b) \times \text{Hom}(b, c) \rightarrow \text{Hom}(a, c)$$

A metric space has:

points a, b, \dots

numbers $d(a, b)$

triangle inequality

$$d(a, b) + d(b, c) \geq d(a, c)$$

In fact, both are special cases of the concept of enriched category.

(A metric space is a category enriched in the poset (\mathbb{R}^+, \geq) with $\otimes = +$.)

2. The magnitude of a metric space

The magnitude of a finite metric space (concretely)

Starting from the general definition of the magnitude of an enriched category, we can specialize to metric spaces. Here's what we get.

To compute the magnitude of a finite metric space $A = \{a_1, \dots, a_n\}$:

- write down the $n \times n$ matrix with (i, j) -entry $e^{-d(a_i, a_j)}$
- invert it
- add up all n^2 entries.

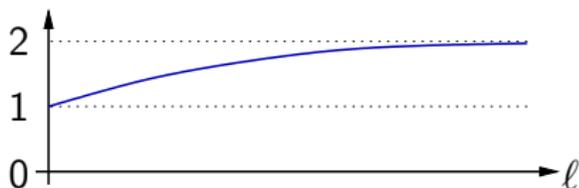
And that's the magnitude $|A|$.

(Where does the “ e^{-x} ” come from?

It's because $f: x \mapsto e^{-x}$ is essentially the only function satisfying $f(x + y) = f(x) \cdot f(y)$.)

The magnitude of a finite metric space: first examples

- $|\emptyset| = 0$.
- $|\bullet| = 1$.
- $|\overset{\leftarrow \ell}{\bullet} \rightarrow \bullet| = \text{sum of entries of } \begin{pmatrix} e^{-0} & e^{-\ell} \\ e^{-\ell} & e^{-0} \end{pmatrix}^{-1} = \frac{2}{1 + e^{-\ell}}$



- If $d(a, b) = \infty$ for all $a \neq b$ then $|A| = \text{cardinality}(A)$.

Slogan: Magnitude is the 'effective number of points'

Magnitude functions

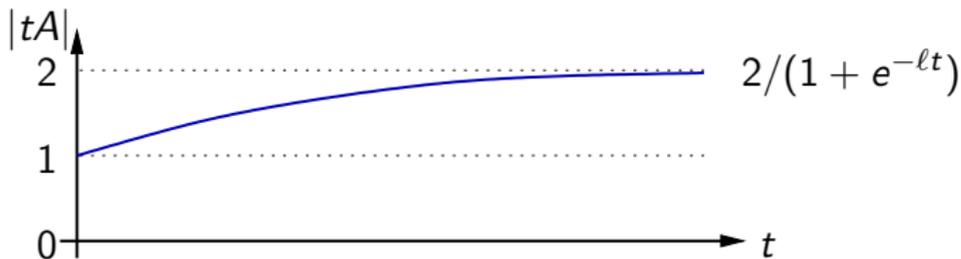
Magnitude assigns to each metric space not just a *number*, but a *function*.

For $t > 0$, write tA for A scaled up by a factor of t .

The **magnitude function** of a metric space A is the partial function

$$\begin{aligned} \mathbb{R}^+ &\rightarrow \mathbb{R} \\ t &\mapsto |tA|. \end{aligned}$$

E.g.: the magnitude function of $A = (\bullet \xleftarrow{\ell} \bullet \xrightarrow{\ell} \bullet)$ is



A magnitude function has only finitely many singularities (none if $A \subseteq \mathbb{R}^n$).

It is increasing for $t \gg 0$, and $\lim_{t \rightarrow \infty} |tA| = \text{cardinality}(A)$.

The magnitude of a compact metric space

In principle, magnitude is only defined for enriched categories *with finitely many objects* — here, *finite* metric spaces.

Can the definition be extended to *compact* metric spaces?



Theorem (Mark Meckes)

All sensible ways of extending the definition of magnitude from finite metric spaces to compact 'positive definite' spaces are equivalent.

Proof Uses some functional analysis.

Positive definite spaces include all subspaces of \mathbb{R}^n with Euclidean or ℓ^1 (taxicab) metric, and many other common spaces.

The **magnitude** of a compact positive definite space A is

$$|A| = \sup\{|B| : \text{finite } B \subseteq A\}.$$

First examples

E.g. Line segment: $|t[0, \ell]| = 1 + \frac{1}{2}\ell \cdot t$.

E.g. Let $A \subseteq \mathbb{R}^2$ be an axis-parallel rectangle with the ℓ^1 (taxicab) metric.
Then

$$|tA| = \chi(A) + \frac{1}{4}\text{perimeter}(A) \cdot t + \frac{1}{4}\text{area}(A) \cdot t^2.$$

0-dimensional measure

1-dimensional measure

2-dimensional measure

dimension of A

Magnitude encodes geometric information

Theorem (Meckes) *Let A be a compact subset of \mathbb{R}^n , with Euclidean metric.*

*From the magnitude function of A , you can recover its **Minkowski dimension**.*

Proof Uses a deep theorem from potential analysis, plus the notion of maximum diversity.

Theorem (Willerton) *Let A be a homogeneous Riemannian n -manifold. Then as $t \rightarrow \infty$,*

$$|tA| = a_n \text{vol}(A) \cdot t^n + b_n \text{tsc}(A) \cdot t^{n-2} + O(t^{n-4}),$$

*where a_n and b_n are constants and **tsc** is total scalar curvature.*

Proof Uses some asymptotic analysis.

Magnitude encodes geometric information



Theorem (Barceló and Carbery) *From the magnitude function of $A \subseteq \mathbb{R}^n$, you can recover the **volume** of A .*

Proof Uses PDEs and Fourier analysis.

Theorem (Barceló and Carbery) *For odd n , the magnitude function of the Euclidean ball B^n is a rational function over \mathbb{Q} .*

Examples

$$|tB^1| = 1 + t$$

$$|tB^3| = 1 + 2t + t^2 + \frac{1}{6}t^3$$

$$|tB^5| = \frac{360 + 1080t + 1080t^2 + 525t^3 + 135t^4 + 18t^5 + t^6}{120(3 + t)}$$

Magnitude encodes geometric information



Theorem (Gimperlein, Goffeng and Louca)
*Let A be a sufficiently regular subset of \mathbb{R}^n . From the magnitude function of A , you can recover the **surface area** of A .*

Proof Uses heat trace asymptotics (techniques related to heat equation proof of Atiyah–Singer index theorem) and treats t as a *complex* parameter.

Theorem (Gimperlein and Goffeng) *Let A and B be nice subsets of \mathbb{R}^n . Then*

$$|t(A \cup B)| + |t(A \cap B)| - |tA| - |tB| \rightarrow 0$$

as $t \rightarrow \infty$.

Magnitude of metric spaces doesn't *literally* obey inclusion-exclusion, as that would make it trivial. But it *asymptotically* does.

$2\frac{1}{2}$. *The magnitude of a graph*

The magnitude of a graph

Graph will mean finite, undirected graph with no multiple edges or loops.

Let G be a graph, with vertices a_1, \dots, a_n .

The **distance** between two vertices is the shortest path-length between them.

We will use a formal variable q . (Think of it as e^{-t} .)

Let Z_G be the $n \times n$ matrix with (i, j) -entry $q^{d(a_i, a_j)}$.

Then Z_G is invertible over the field $\mathbb{Q}(q)$ of rational functions in q .

The **magnitude** of G is $|G| = |Z_G| \in \mathbb{Q}(q)$.

E.g.:

$$\left| \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \end{array} \right| = \frac{5 + 5q - 4q^2}{(1 + q)(1 + 2q)} = 5 - 10q + 16q^2 - 28q^3 + \dots$$

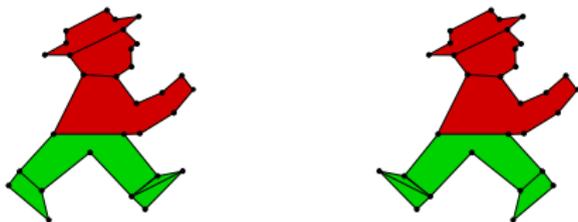
The magnitude of a graph: properties

Cardinality-like properties

- $|\bullet \bullet \bullet| = 3$, etc.
- $|G \square H| = |G| \cdot |H|$, where \square is the 'cartesian product' of graphs
- $|G \cup H| = |G| + |H| - |G \cap H|$, under hypotheses.

Magnitude also bears some resemblance to the Tutte polynomial.

For instance, these two graphs have the same magnitude:



This is a **Whitney twist**. Their invariance under magnitude has been studied, clarified and generalized by Emily Roff.



But neither magnitude nor the Tutte polynomial is determined by the other.

3. *Magnitude homology*

The idea in brief

Find a homology theory for enriched categories that categorifies magnitude.

This was first done for graphs (seen as metric spaces) by Hepworth and Willerton in 2015: given a graph G ,



- they defined a group $H_{n,\ell}(G)$ for all integers $n, \ell \geq 0$ (a *graded* homology theory);
- writing $\chi_\ell(G) = \sum_n (-1)^n \text{rank}(H_{n,\ell}(G))$, they showed that the generating function

$$t \mapsto \sum_{\ell} \chi_\ell(G) q^\ell$$

is equal to the magnitude of G .

So: *the Euler characteristic of magnitude homology is magnitude.*

From graphs to enriched categories and metric spaces



The definition of magnitude homology was extended from graphs to enriched categories in work with Mike Shulman. Definition omitted!

It also generalizes the homology of ordinary categories.

Metric spaces are enriched categories, so we get a *homology theory of metric spaces*.

It is an \mathbb{R}^+ -graded homology theory: for each metric space A , integer $n \geq 0$ and real $\ell \geq 0$, there is an abelian group $H_{n,\ell}(A)$.

For *finite* metric spaces, magnitude homology categorifies magnitude:

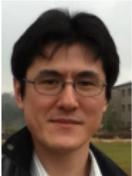
$$|tA| = \sum_{\ell \in \mathbb{R}^+} \chi_\ell(A) e^{-\ell t},$$

where $\chi_\ell(A) = \sum_n (-1)^n \text{rank}(H_{n,\ell}(A))$ as before.

But there is no such theorem for non-finite spaces!

This is the major challenge in the field.

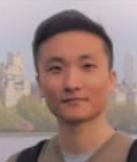
Magnitude homology of metric spaces

-  While ordinary homology detects the *existence* of holes, magnitude homology detects the *diameter* of holes (Ryuki Kaneta and Masahiko Yoshinaga).
- Magnitude homology can distinguish between graphs that have the same magnitude (Yuzhou Gu) 
-  The magnitude homology of a convex subset of \mathbb{R}^n is trivial (Kaneta and Yoshinaga; Benoît Jubin).
- If a metric space contains a closed geodesic then its 2nd magnitude homology group is nontrivial (Yasuhiko Asao) 
- A slogan of Kiyonori Gomi:



The more geodesics are unique,
the more magnitude homology is trivial

What's happening in magnitude homology?

-   There is a relationship between magnitude homology and *persistent homology*—but they detect different information (Nina Otter; Simon Cho).
- Magnitude homology of *random graphs*, and applications to *networks* (Giuliamaria Menara) 
- A theory of magnitude *cohomology* (Hepworth).
- A *spectral sequence* uniting magnitude homology and path homology (Asao; Hepworth and Willerton; Gomi; Roff).
-  A concept of *magnitude homotopy* (Yu Tajima and Yoshinaga).
- Theorems on *magnitude homology equivalence* of subsets of \mathbb{R}^n , involving convex geometry (with Adrián Doña Mateo) 
- And lots more!

Snapshots of a few applications

Magnitude Distance: A Geometric Measure of Dataset Similarity

Sahel Torkamani

Henry Gouk

Rik Sarkar

Snapshots of a few applications

Magnitude Distance: A Geometric Measure of Dataset Similarity

Geometry-Aware Edge Pooling for Graph Neural Networks

Katharina Limbeck^{*,1,2} **Lydia Mezrag**^{*,3,4} **Guy Wolf**^{†,3,4} **Bastian Rieck**^{†,1}

elmholtz Munich

²Technical University of Munich

³Université de Mo

⁴Mila - Quebec AI Institute

⁵Université de Fribourg

Snapshots of a few applications

**Magnitude Distance: A Geometric Measure
of Dataset Similarity**

**Geometry-Aware Edge Pooling
for Graph Neural Networks**

THE MAGNITUDE OF CATEGORIES OF TEXTS ENRICHED BY
LANGUAGE MODELS

Kat

elmh

TAI-DANAE BRADLEY AND JUAN PABLO VIGNEAUX

Snapshots of a few applications

Magnitude Distance: A Geometric Measure of Dataset Similarity

Geometry-Aware Edge Pooling for Graph Neural Networks

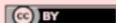
THE MAGNITUDE OF CATEGORIES OF TEXTS ENRICHED BY LANGUAGE MODELS

Kat

elmh

ROYAL SOCIETY
OPEN SCIENCE

royalsocietypublishing.org/journal/rsos



Topological magnitude for protein flexibility analysis

Wanying Bi^{1,2,3}, Hongsong Feng⁴, Jie Wu⁵, Jingyan Li⁵
and Guo-Wei Wei²

Snapshots of a few applications

Magnitude Distance: A Geometric Measure of Dataset Similarity

Geometry-Aware Edge Pooling for Graph Neural Networks

THE MAGNITUDE OF CATEGORIES OF TEXTS ENRICHED BY LANGUAGE MODELS

Kat

elmh

ROYAL SOCIETY

Topological magnitude for protein flexibility analysis

Wanying Bi^{1,2,3}, Hongsong Feng⁴, Jie Wu⁵, Jingyan Li⁵ and Guo-Wei Wei²

 frontiers | Frontiers in [Psychiatry](#)

Blurred magnitude homology of functional connectome for ASD diagnosis

Alexander Kachura^{1*}, Vsevolod Chernyshev², Oleg Kachan¹ and Egor Levchenko³

4. *(Bio)diversity*

What is diversity?

Conceptual question Given an ecological community, consisting of individuals grouped into species, how can we reasonably quantify its 'diversity'?

Simplest answer Count the number n of species present.

Mathematically: cardinality of a finite set.

Better answer Use the relative abundance distribution $\mathbf{p} = (p_1, \dots, p_n)$ of species. ("Relative" means $\sum p_i = 1$.)

For any choice of parameter $q \in \mathbb{R}^+$, we can quantify diversity as

$$D_q(\mathbf{p}) = \left(\sum_i p_i^q \right)^{1/(1-q)}.$$

E.g. If $\mathbf{p} = (\frac{1}{n}, \dots, \frac{1}{n})$ then $D_q(\mathbf{p}) = n$.

We've defined *infinitely many* diversity measures here, one for each $q \in \mathbb{R}^+$.

Mathematically: \approx entropy of a probability distribution on a finite set.

Similarity between species

Even better answer Also use a matrix Z of similarities between species.

The idea: Z_{ij} is the similarity between species i and species j .

Interpretation:

- If $Z_{ij} = 0$, species i and j are completely dissimilar—nothing in common.
- If $Z_{ij} = 1$, species i and j are identical. (So normally $Z_{ii} = 1$.)

This gives an $n \times n$ similarity matrix $Z = (Z_{ij})_{i,j=1}^n$.

How do we quantify similarity?

However we like! Examples:

- The “naive model”, where distinct species have nothing in common: Z is the identity matrix I .
- Genetically, phylogenetically, functionally, morphologically, ...
- If we have a metric d on the set $\{1, \dots, n\}$ of species, we can define the similarities by $Z_{ij} = e^{-d(i,j)}$.

Species-sensitive diversity measures

Take an ecological community with similarity matrix Z and a relative abundance distribution \mathbf{p} .

For any choice of parameter $q \in \mathbb{R}^+$, we can quantify diversity as

$$D_q^Z(\mathbf{p}) = \left(\sum_i p_i (Z\mathbf{p})_i^{q-1} \right)^{1/(1-q)}.$$

The formula is not important here. But...



Discovery (with Christina Cobbold) Most of the biodiversity measures most commonly used in ecology are special cases of D_q^Z .

Mathematically: \approx entropy of a probability distribution on a finite metric space.

The maximization problem

Fix a list of species, with known similarity matrix Z .

What is the maximum diversity that can be achieved by varying the species abundances? In other words: what is $\sup_{\mathbf{p}} D_q^Z(\mathbf{p})$?

In principle, the answer depends on the parameter q . But...

Theorem (with Meckes) *The answer is independent of q .*

So, $\sup_{\mathbf{p}} D_q^Z(\mathbf{p})$ is a well-defined number associated with the matrix Z — the **maximum diversity** $D_{\max}(Z)$ of Z .

Fact $D_{\max}(Z)$ is the magnitude of a certain submatrix of Z .

Conclusion: Magnitude is closely related to maximum diversity.

Back to geometry

The diversity measures D_q^Z and the maximum diversity theorem can be extended from finite to *compact* spaces (joint theorem with Roff):

$\sup_{\mathbf{p}} D_q^Z(\mathbf{p})$ is independent of q .

So, every compact metric space A has a **maximum diversity** $D_{\max}(A) \in \mathbb{R}^+$.

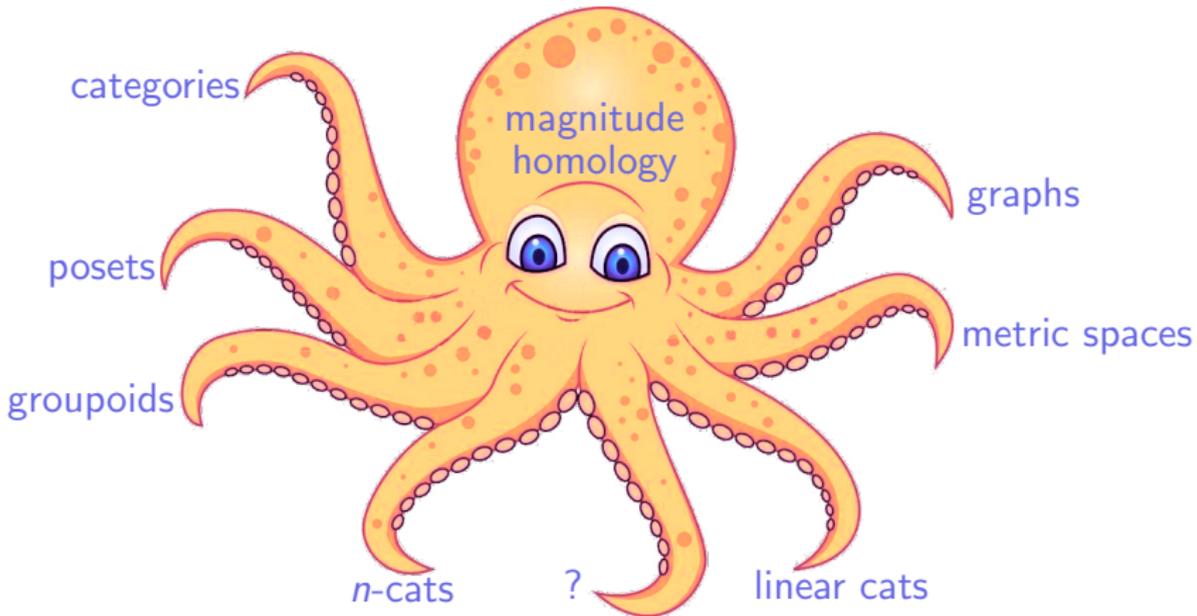
It is equal to the magnitude of a certain closed subset of A .

We know some things about the behaviour of maximum diversity.

For example, Meckes showed that $D_{\max}(tA)$ grows like $t^{\dim A}$ as $t \rightarrow \infty$, where **dim A** is Minkowski dimension.

But we don't know the maximum diversity of even some very simple spaces, such as the 2-dimensional Euclidean disc!

Summary



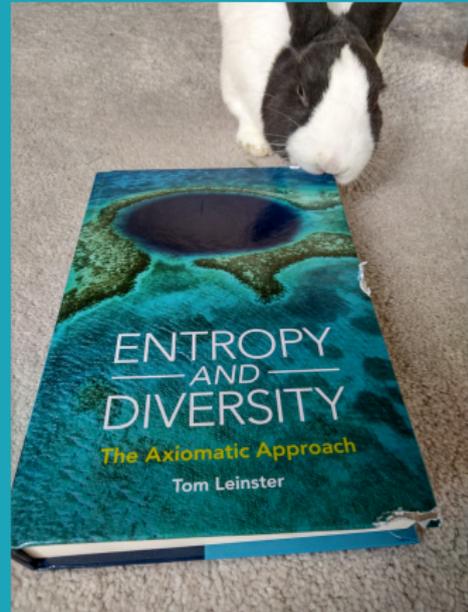
References



The magnitude bibliography:

www.maths.ed.ac.uk/~tl/magbib

Lists all publications on magnitude.



The diversity story