Some applications of polynomial functors

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Outline

1 **Introduction**
- Why **Poly**?
- Today's talk

2 Functional programming

3 Dynamical systems

4 Databases

5 Conclusion

All roads lead to Rome; what did Rome have??

The Polynomial Functors workshops were a confluence of researchers.

- Marcelo Fiore, Steve Awodey, Thorsten Altenkirsch: type theory
- Fred Norvall, Exequiel Rivas, Paul Taylor: programming languages
- David Spivak: database theory and dynamical systems
- Eric Finster, PL Curien, Kristina Sojakova, David Gepner: ∞-caty's
- Todd Trimble, André Joyal, Tarmo Uustalu: new theory about **Poly**
- Helle Hvid Hansen, Sean Moss, Bart Jacobs: Logic
- Brandon Shapiro, Michael Batanin: polynomial monads for formal CT
- And many more... Ross Street, etc., etc.

What do these fields have in common?

- What are polynomial functors *about*?
- What makes polynomial functors a center for *this kind* of convergence?

Why Poly?

"Why" does **Poly** have such centrality within category theory?

- I don't know why it applies to so many things.
- But I do know that categorically, it is incredibly rich and well-behaved:
  - Coproducts and products that agree with usual polynomial arithmetic;
  - All limits and colimits;
  - At least three orthogonal factorization systems;
  - A symmetric monoidal structure $\otimes$ distributing over $+$;
  - A cartesian closure q^p and monoidal closure $[p, q]$ for $\otimes$;
  - Another nonsymmetric monoidal structure $\triangleleft$ that's duoidal with $\otimes$;
  - A left $\triangleleft$-coclosure $\left[\begin{smallmatrix}-\\-\end{smallmatrix}\right]$, meaning $\mathbf{Poly}(p, q \triangleleft r) \cong \mathbf{Poly}(\left[\begin{smallmatrix}r\\p\end{smallmatrix}\right], q)$;
  - An indexed right $\triangleleft$-coclosure (Myers?), i.e. $\mathbf{Poly}(p, q \triangleleft r) \cong \sum_{f\colon\ p(1)\to q(1)} \mathbf{Poly}(p \overset{f}{\frown} q, r)$;
  - An indexed right $\otimes$-coclosure (Niu?), i.e. $\mathbf{Poly}(p, q \otimes r) \cong \sum_{f\colon\ p(1)\to q(1)} \mathbf{Poly}(p \overset{f}{\nearrow} q, r)$;
  - At least eight monoidal structures in total;
  - $\triangleleft$-monoids generalize Σ-free operads;
  - $\triangleleft$-comonoids are exactly categories; bicomodules are data migrations. This is $\mathbb{C}\mathbf{at}^\sharp$.
- See "A reference for categ'ical structures on **Poly**", arXiv: 2202.00534

Getting to know Poly: the lens pattern

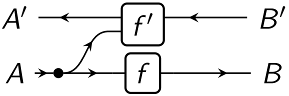
We'll begin with the subject of a lot of recent ACT attention: *lenses*.

Definition

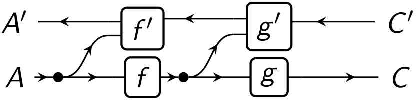
There is a category **Lens** whose objects are pairs of sets

$$\mathrm{Ob}(\mathbf{Lens}) := \mathrm{Ob}(\mathbf{Set} \times \mathbf{Set}), \quad \text{denoted } \begin{bmatrix} A' \\ A \end{bmatrix}$$

and for which a morphism $\begin{bmatrix} A' \\ A \end{bmatrix} \to \begin{bmatrix} B' \\ B \end{bmatrix}$ consists of a pair (f, f') where



i.e. $f\colon A \to B$ and $f'\colon A \times B' \to A'$. Composition is:



Understanding the lens pattern

There are many examples of the lens pattern: namely in

- functional programming, ✓
- open dynamical systems, ✓
- wiring diagrams, ✓
- deep learning, no time today
- open games, no time today and
- databases. ✓

We can understand **Lens** $\left(\begin{bmatrix} A' \\ A \end{bmatrix}, \begin{bmatrix} B' \\ B \end{bmatrix}\right)$ in terms of *polynomial functors*.

Polynomial functors

A functor $p\colon \mathbf{Set} \to \mathbf{Set}$ is *polynomial* if it is a coproduct of representables.

- Taking all natural transformations as maps, we get a category **Poly**.
- I denote objects in it like this: $p := y^5 + 3y^2 + 7$.
- For example, $p(0) \cong 7$, $p(1) \cong 11$, and $p(2) \cong 51$.
- Let's call p a *monomial* if it is of the form $p \cong Ay^{A'}$, e.g. $5y^{73}$.

Theorem

There is an isomorphism of categories

$$\mathbf{Lens} \cong \mathbf{Poly}_{Monomial}$$

where $\mathbf{Poly}_{Monomial}$ *is the full subcategory spanned by the monomials.*

In other words, a **Poly** map $Ay^{A'} \to By^{B'}$ is a **Lens** map $\begin{bmatrix} A' \\ A \end{bmatrix} \to \begin{bmatrix} B' \\ B \end{bmatrix}$.

Today: introduce Poly in terms of its applications

Davide asked me to speak mainly about the applications of **Poly**.

- There are many, including to pure math.
- I'll focus on a few: programming, dynamical systems, databases.
- I'll introduce structures of **Poly** as we go.

Outline

1 Introduction

2 **Functional programming**
- Polymorphic data types
- Deeper look at **Poly**
- Algebraic datatypes

3 Dynamical systems

4 Databases

5 Conclusion

Polymorphic data types and maps

In functional languages such as Haskell, you often see things like this:

```
data Foo y = Bar y y y | Baz y y | Qux | Quux
data Maybe y = Just y | Nothing
```

- These are polynomials: $y^3 + y^2 + 2$ and $y + 1$ respectively.
- They're "polymorphic" in that
  - they act on any Haskell type `Y` in place of the variable `y`, and
  - for any map `f : Y1 -> Y2` there's a map `Foo Y1 -> Foo Y2`

What is a natural transformation `Corge: Foo` $\rightsquigarrow$ `Maybe`?

- To each type constructor (`Bar`, `Baz`, `Qux`, `Quux`) in `Foo` ...
- ... it assigns a type constructor (`Just` or `Nothing`) in `Maybe`,...
- ... and a way to grab as many `y`'s as `Maybe` needs from `Foo`'s term.

There are 12=6+3+2+1 ways to do it. Three examples:

```
Corge (Bar a b c)=Just a;  Corge (Baz a b)=Just a; Corge Qux=Nothing; Corge Quux=Nothing
Corge (Bar a b c)=Just b;  Corge (Baz a b)=Just a; Corge Qux=Nothing; Corge Quux=Nothing
Corge (Bar a b c)=Nothing; Corge (Baz a b)=Just b; Corge Qux=Nothing; Corge Quux=Nothing
```

Deeper look at objects and morphisms in Poly

Let's slow down and understand **Poly** a little better.

- A representable functor **Set** $\to$ **Set** is one of the form

$$y^A := \mathbf{Set}(A, -)$$

  for example y^2 takes any set Y to $Y \times Y$.
- y^1 is isomorphic to the identity, and y^0 is constant 1.
- A polynomial functor is a coproduct of representables

$$p := \sum_{i \in I} y^{p[i]}$$

  Note that $I \cong p(1)$, so we write $p := \sum_{i \in p(1)} y^{p[i]}$.

Maps $p \to q$ are computed using Yoneda and univ. property of coproducts.

$$\begin{aligned}\mathbf{Poly}(p, q) &= \mathbf{Poly}\Big(\sum_{i \in p(1)} y^{p[i]}, \sum_{j \in q(1)} y^{q[j]}\Big)\\ &\cong \prod_{i \in p(1)} \sum_{j \in q(1)} \mathbf{Set}(q[j], p[i])\end{aligned}$$

Unpacking in the Haskell case

That might be daunting, but it's pretty easy when you get used to it.

- Let's see another example of a natural transformation.
- Here are two polynomial datatypes, $p := y^3 + y$ and $q := 2y^2 + 1$.

```
data p y = pFoo y y y | pBar y
data q y = qFoo y y   | qBar y y | qBaz
```

- What's a natural transformation `Corge:  p ⇝ q`?

This crazy formula $\mathbf{Poly}(p, q) = \prod_{i \in p(1)} \sum_{j \in q(1)} \mathbf{Set}(q[j], p[i])$ says:

- For each $i \in p(1)$, namely `pFoo` and `pBar`, we need to …
- … choose $j \in q(1)$, namely either `qFoo`, `qBar`, or `qBaz` and then …
- … for each variable there in q, choose one of the variables in p.

```
Corge :  forall y.  p y -> q y
Corge pFoo (a b c) = qBar (b a)  -- Corge is one of
Corge pBar (a)     = qFoo (a a), -- 57 possible maps.
```

Algebraic datatypes

Another thing you see in Haskell is something like this:

```
List a = Nil | Cons a (List a)
```

For some type `a`, e.g. `a = Int`. What is going on here?

- This is called an *algebraic data type*.
- It looks like `List a` is being defined recursively, in terms of itself.
- But we can break it into two pieces: a functor and its fixed points.

```
ListF a y = Nil | Cons a y
```

This is the polynomial $p_A \coloneqq 1 + Ay$ for some set $A \in \mathbf{Set}$. (I like my sets capitalized.)

- Polynomial functors have initial algebras and final coalgebras.
  - That is, there is an initial $S \in \mathbf{Set}$ equipped with $p(S) \to S$.
  - And there is a final $T \in \mathbf{Set}$ equipped with $T \to p(T)$.
- The initial algebra of p_A is carried by $\sum_{n\in\mathbb{N}} A^n$, classic lists.
- The terminal coalgebra of p_A is carried by $A^{\mathbb{N}} + \sum_{n\in\mathbb{N}} A^n$, streams.

Outline

1. Introduction
2. Functional programming
3. **Dynamical systems**
   - Wiring diagrams and interaction patterns
4. Databases
5. Conclusion

Various notions of dynamical system

Moving on, there are many reasonable definitions of dynamical system.

- Fix a monoid $(T, 0, +)$. Then a T-Dyn. Sys. is a T-action on $S \in$ **Set**.
- For example, an action $\mathbb{R} \times S \to S$ let's you evolve s by any $t \in \mathbb{R}$.
- We'll briefly return to this sort later, but it's not quite satisfactory.
- I want open dynamical systems, ones that can interact with others.

$$A \;-\!\!\boxed{S^{\circlearrowleft}}\!\!-\; B$$

Let A, B be sets or spaces. Notions of (A, B)-dynamical systems include:

- System of ODEs, parameterized by A and reading out B's.
- Moore machine: a set S and functions $r\colon S \to B$ and $u\colon A \times S \to S$.
- Mealy machine: a set S and a function $f\colon A \times S \to S \times B$.

Dynamical systems in terms of Poly

Let's discuss each of these (saving the monoid action for later).

- For any manifold M, let TM be its tangent bundle.
  - At every point $m \in M$, we have a tangent space $T_m M$.
  - For example, if $M = \mathbb{R}^n$ then $TM \cong \mathbb{R}^n \times \mathbb{R}^n$ and $T_m M \cong \mathbb{R}^n$.
- Then an A-parameterized system of ODEs reading out B's is a map:

$$\varphi\colon \sum_{m \in M} y^{T_m M} \to By^A$$

  Let's think of M as the state space. Then
  - for each $m \in M$, we get a readout $\varphi_1(m)$ and ...
  - for each $a \in A$, we get a tangent vector $\varphi^\sharp(m, a) \in T_m M$.

(A, B)-Moore machines are easier.

- A set S and functions $r\colon S \to B$ and $u\colon S \times A \to S$?
- That's the same data a map of polynomials $Sy^S \to By^A$.
- It's also the same as a By^A coalgebra: $S \to BS^A$.

Mealy machines

The difference between Moore and Mealy machines involves instantaneity.

- An (A, B)-Moore machine is $S \to B$ and $A \times S \to S$.
- An (A, B)-Mealy machine is $A \times S \to B$ and $A \times S \to S$.
  - In Mealy, the input A can immediately affect the output B.
  - A Moore machine can be regarded as a Mealy machine (drop A).

It took me a long time to realize that the converse is also true.

- An (A, B)-Mealy machine is an (A, B^A)-Moore machine.
- Indeed, that's $S \to B^A$ and $A \times S \to S$.
- A Mealy machine is a Moore machine that outputs functions.

The transformation isn't out of the blue: it comes from monoidal closure.

Monoidal closure of Poly

Poly has a monoidal closed structure $(y, \otimes, [-,-])$.

- Let $p := \sum_{i \in p(1)} y^{p[i]}$ and $q := \sum_{j \in q(1)} y^{q[j]}$
- The *Dirichlet product* $p \otimes q$ has monoidal unit y and is given by:

  $$p \otimes q := \sum_{(i,j) \in p(1) \times q(1)} y^{p[i] \times q[j]}$$

  We'll use that on the next slide.

- It has an internal hom $[p, q]$, given by

  $$[p, q] := \sum_{\varphi \colon p \to q} y^{\sum_{i \in p(1)} q[\varphi_1 i]}$$

  That's a lot to take in, so let's try it for $p := Ay^B$ and $q := y$.
  - First, a map $\varphi \colon Ay^B \to y$ is just a function $A \to B$.
  - Since $p(1) = A$ and $q[!] = 1$, we have $[Ay^B, y] = B^A y^A \cong (By)^A$.

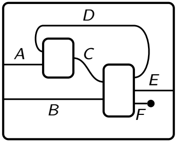
So an $[Ay^B, y]$-coalgebra $S \to (BS)^A$ is an (A, B)-Mealy machine.

Wiring diagrams

Let's depict monomials By^A as boxes with A-inputs and B-outputs:

By^A is depicted 

Here's a picture of a kind of *interaction pattern* called a wiring diagram:

$\varphi :=$ 

It has two inner boxes and one outer box, and represents a map

$$\varphi\colon Cy^{AD} \otimes DEFy^{BC} \to Ey^{AB}$$

In other words the picture tells us about two functions:

$$C(DEF) \to E \qquad \text{and} \qquad C(DEF)(AB) \to (AD)(BC)$$

Wiring diagrams allow projection, splitting, and permuting variables.

More general interfaces

A polynomial $p = \sum_{i \in p(1)} y^{p[i]}$ can be understood as an interface that

- outputs "positions" $i \in p(1)$ and
- inputs "directions" $d \in p[i]$ that can depend on its position.
- So By^A can output elements of B and input elements of A.
- But $y^2 + y$ is like an eyeball: its positions are open and closed and...
- ... when it's open it receives a bit; when it's closed it receives no bits.

An *clocked interaction pattern* of interfaces $p_1, \ldots p_k$ inside p' is a map

$$\varphi \colon p_1 \otimes \cdots \otimes p_k \to p'$$

A wiring diagram is a very special case. For example, there is only one WD

$$2y^3 \otimes 3y^5 \to 2y^5$$

but there are $2^6 * 15^{30} \approx 10^{37}$ clocked interaction patterns.

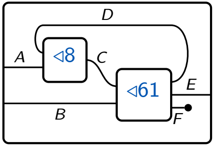
Composition in Poly: removing the clock

Composing polynomials is a monoidal operation $\triangleleft$: **Poly** $\times$ **Poly** $\to$ **Poly**.

- I denote this functor by $\triangleleft$, leaving $\circ$ for composition of morphisms.
- It is straightforward, e.g. $y^2 \triangleleft (y+1) \cong y^2 + 2y + 1$. The unit is y.

You can use this to make dynamical systems run faster.

- Any map $Sy^S \to p$ induces $Sy^S \to p^{\triangleleft n}$ for any n.
- Because there's a certain semi-monad structure on Ay^B, ...
- ...we can run interior boxes at n-times speed for any $n \geq 1$.

$\varphi :=$ 

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Categorical databases

A database is a collection of tables whose columns can refer to other tables.

- One way to conceptualize this is as a category $\mathcal{C}$, "the schema"...
- ... together with a functor (copresheaf) $D\colon \mathcal{C} \to \mathbf{Set}$, "the keys"...
- ... and one of many possible ways to categorically handle "attributes".
- This approach to databases has been implemented several times.

The two things one does with databases are: migrate and aggregate.

- Data migration means moving data from one schema to another.
- It includes querying: asking for all matches for a certain pattern.
- Aggregation means accumulating attribute values over a column...
- ... where we assume that the attribute has a comm. monoid structure.

All of this fits nicely into the **Poly** ecosystem.

Comonoids and bicomodules in Poly

By a theorem of Shulman, comonoids in $(\mathbf{Poly}, y, \triangleleft)$ form an equipment.

- By theorems of Ahman-Uustalu and Garner, it has relevant semantics.
- Its objects are exactly categories, so I call it $\mathbb{C}\mathbf{at}^\sharp$.
- Its horizontal maps generalize both copresheaves and data migration.
- The subcategory carried by linear polynomials is exactly $\mathbb{S}\mathbf{pan}$.
- It contains Gambino-Kock's $\mathbf{PolyFun}_{\mathbf{Set}}$ as a full sub equipment.
- It's got local monoidal closed structures, and tons of other structure.

You can define not only data migration but also aggregation in this setting.

- To do so requires all the structures we've discussed so far.
- For example, it turns out that the operation of transposing a span...
- ... can be split up into two more primitive universal operations.

Finally, keeping an old promise...

- The vertical maps are in $\mathbb{C}\mathbf{at}^\sharp$ are called cofunctors.
- If y^T is a monoid, then a cofunctor $Sy^S \to y^T$ is a T-action on S.
- Using cofree comonoids, dyn. systems are subsumed as "databases".

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- Summary

Summary

The polynomial ecosystem is very rich.

- It's got an abundance of structure; that's difficult to over-state.
  - I now know of eight different monoidal structures on **Poly**.
  - How many structures are we still missing?
- **Poly** offers a single setting in which lots of ACT subjects live.
  - Programming, dynam'l systems, databases, deep learning, games.
  - But how do they come together? How should they *interact*?

There's ton's to do; please join in the fun!

Thanks! Comments and questions welcome...

Adaptive interaction patterns

We want to remove the fixed nature of interaction patterns.

- That is, we want wiring pattern itself to change through time.
- We might call this "adapting"; we'll briefly consider "goals" on p. 22.

Given interfaces $p_1, \dots, p_k$ and p', we want a changing interaction pattern.

- Let $p := p_1 \otimes \cdots \otimes p_k$ and recall the internal hom

$$[p, p'] \cong \sum_{\varphi \colon p \to p'} y^{\sum_{i \in p(1)} p'[\varphi_1 i]}.$$

- Its positions are interaction patterns $\varphi \colon p_1 \otimes \cdots \otimes p_k \to p'$!
- And a direction at φ is "the data flowing on all the wires".
- For example if $p_i = B_i y^{A_i}$ then direction set is always $B_1 \cdots B_k A'$.

So a $[p, p']$-coalgebra is a Moore machine:

- it outputs interaction patterns and updates based on what's flowing.
- Define a category-enriched operad $\mathbb{O}\mathbf{rg}$ with objects Ob(**Poly**) and...
- ... hom-caty's $[p_1 \otimes \cdots \otimes p_k, p']$-**Coalg**, or $[\mathfrak{c}_{p_1} \otimes \cdots \otimes \mathfrak{c}_{p_k}, p']$-**Coalg**.
- This is the subject of a paper called *Learners' languages*.

Deep learning falls out

Artificial neural networks are adaptive organizations in the above sense.

- Let $t \coloneqq \sum_{x \in \mathbb{R}} y^{T_x \mathbb{R}}$ be the tangent bundle; note $t^{\otimes n} \cong \sum_{x \in \mathbb{R}^n} y^{T_x \mathbb{R}^n}$.
- A $[t^{\otimes n}, t]$-coalgebra is just a Moore machine with a fancy interface.
  - Let $P \coloneqq \mathbb{R}^{n+1}$; think of $(b, w_1, \ldots, w_n) \in P$ as bias & weights.
  - Then an artificial neuron is a coalgebra $P \to [t^{\otimes n}, t] \triangleleft P$.
  - For every parameter, we get both a map $\mathbb{R}^n \to \mathbb{R}$ and ...
  - ... a way to convert any tangent vector on $\mathbb{R}$ (loss)...
  - ... to a tangent vector on $\mathbb{R}^n$ (back propagation) ...
  - ... as well as a new parameter (by gradient descent).
- The composite of coalgebras in $\mathbb{O}rg$ runs the DNN as usual.
- Weight tying (as in convolution, recurrent, etc.) is as in Backprop AF.