Categorical algebraic geometry 2-torial

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Overview

There are many approaches to algebraic geometry, and many different ways to arrive at the subject. Here we're going to take an incredibly specific and biased approach: what if you already love 2-categories and want to be able to say the phrase "fpqc sheaf" as quickly as possible, but not *too* quickly? In this series of exercises we will build towards an understanding of **relative algebraic geometry**, which allows us to work in arbitrary (nice) symmetric monoidal categories.

Prerequisites

Quotients of rings, Yoneda embedding, 2-limits, symmetric monoidal categories (cosmoi).

Key concepts

Functor of points, affine scheme, quasi-coherent sheaf, Grothendieck pseudofunctor, pre-topology, faithfully flat topology, algebra over a commutative monoid, sheaf, affine scheme (again).

Further reading

- Bertrand Toën, Michel Vaquié, "Under Spec Z". arXiv:math/0509684
- Bertrand Toen, Gabriele Vezzosi, "Homotopical Algebraic Geometry II: geometric stacks and applications". arXiv:math/0404373

Not exercises

1. Compute the determinant bundle of Ω^1 on $\mathbb{P}^2_{\mathbb{C}}$.

2. Compute the Chern character of a finitely-supported skyscraper sheaf on $\mathbb{P}^1_{\mathbb{C}}$.

3. Let X be a Riemann surface of genus g with canonical bundle K, and L a holomorphic line bundle on X. Prove the Riemann–Roch theorem for Riemann surfaces: $\dim\Gamma(L,X) - \dim\Gamma(L^{-1}\otimes\omega) = \deg L + 1 - g$.

4. Prove that $\mathcal{O}_X$ is coherent if X is Noetherian.

5. Prove that higher direct images of a coherent sheaf are again coherent in the case of a proper morphism between locally Noetherian schemes.

1 Exercises

1.1 Affine schemes as ringed spaces

1. Define the functors $\mathbb{V}$ and $\mathbb{I}$ between **finitely generated finitely presented reduced** commutative k-algebras and **Zariski-closed** subsets of k^n. Observe that they form an adjunction.

2. Let $A = k[x,y]/(x^3+1-y^2)$. "Justify" why the functions "on" $\operatorname{Spec} A = \mathbb{V}(x^3+1-y^2)$ should be given by A itself.

3. Show that $k^\times = k \setminus \{0\}$ is not an **affine scheme** inside k.

4. Show that $k^\times = k \setminus \{0\}$ is an affine scheme inside k^2.

1.2 Affine schemes as functors of points

Some definitions we will explore:

- **affine schemes**: $\mathsf{Aff}_k = \mathsf{CAlg}_k^{\mathrm{op}}$
- **k-spaces**: $\mathsf{Sp}_k = [\mathsf{Aff}^{\mathrm{op}}, \mathsf{Set}]$
- **spectrum functor**: Spec = yoneda.

1. Let A and B be finitely generated (and finitely presented) reduced commutative k-algebras. Explain how morphisms $A \to B$ correspond to solutions of the generators of the **ideal of relations** of A inside B^n for some $n \in \mathbb{N}$. (Consider, for example, the case where $A = k[x,y]/(x^3+1-y^2)$ and $B = k^2$).

2. Use the spectrum functor to give an alternative definition of affine schemes in terms of representability.

3. What is $\operatorname{Spec} A \times_{\operatorname{Spec} S} \operatorname{Spec} B$?

4. Given $A \in \mathsf{CommAlg}$, define $\mathrm{GL}_n(A)$ to be the group of $(n \times n)$ invertible matrices over A. Show that the induced space $\mathrm{GL}_n \colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$ is an affine scheme.

5. Given an affine k-scheme X, write $\mathcal{O}(X)$ to mean the set of k-functions on X. Define $\mathcal{O}(X)$ in terms of a universal property. *Possible hints.* for any morphism $\varphi\colon X \to Y$ we should obtain a restriction function $\varphi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$; a presheaf is determined by morphisms into it from representables; every presheaf is a colimit of representables.

6. Define $\mathbb{A}^n = \operatorname{Spec} k[x_1, \ldots, x_n]$. What is $\mathcal{O}(\mathbb{A}^n)$?

7. Define $\mathbb{G}_\mathrm{m} \colon A \mapsto A^\times$. Prove that $\mathbb{G}_\mathrm{m}$ is affine. What is $\mathcal{O}(\mathbb{G}_\mathrm{m})$?

8. Come up with a conjecture about the composite functor $\mathcal{O} \circ \operatorname{Spec}$. Prove it maybe? Use this to justify why **projective space** $\mathbb{P}^n$ is not affine, if you are told that $\mathcal{O}(\mathbb{P}^n) \cong k$.

1.3 Quasi-coherent sheaves

1. Define $\mathsf{QCoh}_{\text{bad}}(X) = \lim_{\operatorname{Spec} A \to X} \mathsf{Mod}_A$. Describe objects in this category.

2. Define $\mathsf{QCoh}(X) = \underline{\lim}_{\operatorname{Spec} A \to X} \mathsf{Mod}_A$, where $\underline{\lim}$ denotes some notion of 2-limit. Describe how objects in this category differ from those in $\mathsf{QCoh}_{\text{bad}}(X)$.

1.4 The faithfully flat topology associated to a Grothendieck pseudofunctor

Let $M\colon \mathcal{D}^{\text{op}} \to \mathsf{Cat}$ be a **Grothendieck pseudofunctor**.

1. Prove that if $p\colon X \to Y$ in $\mathcal{D}$ is **M-faithfully flat** then p^* is faithful.

2. Prove that M-faithfully flat families define a **Grothendieck pretopology** on $\mathcal{D}$.

1.5 The fpqc topology in relative algebraic geometry

1. Recall the definition of a cosmos, and a module over a commutative monoid in a cosmos.

Let $\mathcal{C}$ be a cosmos. Let $A \in \mathsf{CMon}(\mathcal{C})$, and define the tensor product on A-modules by $X \otimes_A Y = \operatorname{coeq}(X \otimes A \otimes Y \rightrightarrows X \otimes Y)$ and write Mod_A to denote the resulting symmetric monoidal category.

2. Let $A \in \mathsf{CMon}(\mathcal{C})$. Compare the two following putative definitions of **commutative A-algebras**:

   1. $\mathsf{CMon}(\mathsf{Mod}_A)$
   2. $A/\mathsf{CMon}(\mathcal{C})$.

   Which definition should we pick?

3. [Optional] Prove that $\mathsf{CMon}(\mathcal{C})$, $A/\mathsf{CMon}(\mathcal{C})$, and Mod_A are cosmoi.

Redefine $\mathsf{Aff}_{\mathcal{C}}$ as the opposite of the category $\mathsf{CMon}(\mathcal{C})$ of commutative monoids in $\mathcal{C}$. Redefine $\operatorname{Spec} = (-)^{\text{op}}$.

4. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ in $\mathsf{Aff}_{\mathcal{C}}$. Prove that this induces a functor $\mathsf{Mod}_B \to \mathsf{Mod}_A$. Prove that this functor is conservative. Show that this functor has a left adjoint given by $(- \otimes_A B)$.

5. Prove that the pseudofunctor $M\colon (\mathsf{Aff}_{\mathcal{C}}^{\text{op}}) \to \mathsf{Cat}$ that sends $A \to B$ to $\mathsf{Mod}_A \xrightarrow{-\otimes_A B} \mathsf{Mod}_B$ is a Grothendieck pseudofunctor.

The M-faithfully flat topology for the above pseudofunctor is called the **faithfully flat and quasi-compact** (fpqc) topology. Note that this topology is **subcanonical**: all representable presheaves are fpqc sheaves. This can be proved by using the category of descent data, and showing that M is a stack for the fpqc topology. This means that we can (but will not yet) redefine $\mathsf{Aff}_{\mathcal{C}}$ as the category of representable (fpqc) sheaves on $\mathsf{CMon}(\mathcal{C})$.

6. Show that the fpqc topology on $\mathsf{Aff}_{\mathcal{C}}$ is equivalently described by being generated by covering families $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}_{i \in I}$ such that the two following conditions hold:
   - for all $i \in I$, the morphism $\operatorname{Spec} A_i \to \operatorname{Spec} A$ is flat, i.e. such that $- \otimes_A B$ is exact; and
   - there exists some finite subset $J \subseteq I$ such that each $- \otimes_A A_j$ is conservative. (Remark: the fpqc topology is thus quasi-compact)

The following definition is purely to be marvelled at.

Definition. A (Zariski) sheaf X on $\mathsf{Aff}_{\mathcal{C}}$ is a (Zariski) **scheme** (relative to $\mathcal{C}$) if there exists a family $\{X_i\}_{i \in I}$ of affine schemes and a morphism of sheaves $p\colon \coprod_{i \in I} X_i \to X$ such that the two following conditions are satisfied:

1. p is an epimorphism
2. each $X_i \to X$ is a **Zariski open immersion** ("i.e." a flat epimorphism of finite presentation).